

HIDDEN GRASSMANN STRUCTURE IN THE XXZ MODEL V: SINE-GORDON MODEL.

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ABSTRACT. We study one-point functions of the sine-Gordon model on a cylinder. Our approach is based on a fermionic description of the space of descendent fields, developed in our previous works for conformal field theory and the sine-Gordon model on the plane. In the present paper we make an essential addition by giving a connection between various primary fields in terms of yet another kind of fermions. The one-point functions of primary fields and descendants are expressed in terms of a single function $\omega_R^{\text{sG}}(\zeta, \xi|\alpha)$, defined via the data from the thermodynamic Bethe Ansatz equations.

1. INTRODUCTION

The one-point functions are important data in Quantum Field Theory (QFT). Indeed, if one applies the Operator Product Expansion (OPE) for computing ultra-violet asymptotics of correlation functions, two objects are needed: the coefficients of the OPE and the one-point functions. The coefficients of the OPE are purely ultra-violet data which have nothing to do with the infra-red environment of the system. In principle, these coefficients are governed by the convergent perturbation theory based on the ultra-violet Conformal Field Theory (CFT). For the one-point functions the situation is different. These are the data which depend essentially on the infra-red environment, and cannot be obtained from the ultra-violet CFT. So, in order to find them one has to develop methods different from CFT.

Let us illustrate these general ideas on a particular example of two-dimensional Integrable Field Theory (IFT) — the famous sine-Gordon (sG) model. In this paper we consider the sG model not as a perturbation of $c = 1$ CFT, but rather as a perturbation of the complex Liouville model. We emphasise this point of view writing the Euclidean action as

$$(1.1) \quad \mathcal{A}^{\text{sG}} = \int \left\{ \left[\frac{1}{4\pi} \partial_z \varphi(z, \bar{z}) \partial_{\bar{z}} \varphi(z, \bar{z}) - \frac{\mu^2}{\sin \pi \beta^2} e^{-i\beta \varphi(z, \bar{z})} \right] - \frac{\mu^2}{\sin \pi \beta^2} e^{i\beta \varphi(z, \bar{z})} \right\} \frac{dz \wedge d\bar{z}}{2},$$

where the normalisation of the dimensional coupling constant μ^2 is chosen for future convenience. For historical reasons we shall mostly use the parameter

$$(1.2) \quad \nu = 1 - \beta^2$$

as the coupling constant, in term of which the semiclassical domain is $\nu \simeq 1$. The complex Liouville model is conventionally identified with the minimal model of CFT,

and the perturbing field $e^{i\beta\varphi(z,\bar{z})}$ is called $\Phi_{1,3}(z,\bar{z})$. The central charge is

$$c = 1 - 6\frac{\nu^2}{1-\nu}.$$

We shall consider the Euclidean correlation functions on an infinite cylinder of circumference $2\pi R$. We write the point $(z, \bar{z}) = (0, 0)$ simply as 0. The conformal map $w = e^{-z}$ brings the cylinder to the Riemann sphere. We use the symbol $(z, \bar{z}) = -\infty, \infty$ to represent the points corresponding to $w = \infty, 0$ on the latter. We use the name “primary fields” for the exponential fields, which are parametrised as

$$\Phi_\alpha(z, \bar{z}) = e^{\frac{\nu}{2(1-\nu)}\alpha\{i\beta\varphi(z,\bar{z})\}}.$$

Its scaling dimension is $2\Delta_\alpha$ where

$$\Delta_\alpha = \frac{\nu^2}{4(1-\nu)}\alpha(\alpha-2).$$

So, our normalisation is such that the shift of α by $\frac{2(1-\nu)}{\nu}$ corresponds in the CFT language to normal ordered multiplication by the primary field $\Phi_{1,3}$. We consider $\Phi_\alpha(z, \bar{z})$ as a primary field for the model given by the action (1.1). We work with the range

$$0 < \alpha < 2,$$

and when needed, treat physical objects such as correlation functions as analytic continuation from this domain. The analyticity in α is a requirement built implicitly in our definition of the model. The case $\alpha = 0$ is special, and we shall comment on it in subsection 10.3.

The OPE for the product of two primary fields in the sG model looks as follows:

$$(1.3) \quad \Phi_{\alpha_1}(z, \bar{z})\Phi_{\alpha_2}(0) = \sum_{m=-\infty}^{\infty} \sum_{N, \bar{N}} (\mu^2 r^{2\nu})^{|m|} C_{\alpha_1, \alpha_2}^{m, N, \bar{N}} (\mu^4 r^{4\nu}) \\ \times r^{\frac{\nu^2}{1-\nu}\alpha_1\alpha_2 + 2m^2(1-\nu) + 2\alpha m\nu} z^{|N|} \bar{z}^{|\bar{N}|} \mathbf{l}_{-N} \bar{\mathbf{l}}_{-\bar{N}} \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0),$$

where $\alpha = \alpha_1 + \alpha_2$, $r = \sqrt{z\bar{z}}$. The formula (1.3) needs some comments. In order to avoid resonances we impose the requirement of incommensurability: ν , $\nu\alpha_i$ and 1 are linearly independent over \mathbb{Q} . By $\mathbf{l}_{-N} \bar{\mathbf{l}}_{-\bar{N}} \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)$ we mean the unique operator in the sG theory, which tends to the corresponding Virasoro descendant in the conformal limit, and does not contain finite counterterms. Following [1], we use the letters \mathbf{l}_n and $\bar{\mathbf{l}}_n$ to denote the Virasoro generators acting on fields at $(z, \bar{z}) = 0$, to make distinction from the Fourier components of the CFT energy-momentum tensor acting on the states at $(z, \bar{z}) = \pm\infty$. For $N = \{n_1, \dots, n_p\}$, $\bar{N} = \{\bar{n}_1, \dots, \bar{n}_q\}$ with $n_1 \geq \dots \geq n_p > 0$, $\bar{n}_1 \geq \dots \geq \bar{n}_q > 0$, we set

$$\mathbf{l}_{-N} = \mathbf{l}_{-n_1} \cdots \mathbf{l}_{-n_p}, \quad \bar{\mathbf{l}}_{-\bar{N}} = \bar{\mathbf{l}}_{-\bar{n}_1} \cdots \bar{\mathbf{l}}_{-\bar{n}_q}, \quad |N| = \sum_{k=1}^p n_k, \quad |\bar{N}| = \sum_{k=1}^q \bar{n}_k.$$

The structure functions $C_{\alpha_1, \alpha_2}^{m, N, \bar{N}}(t)$ are power series in t . It is assumed [2] that the series converge. But independently on this assumption we shall consider $C_{\alpha_1, \alpha_2}^{m, N, \bar{N}}(t)$

as something known because the coefficients of the series can be expressed through the Coulomb gas integrals.

We want to use the OPE (1.3) for the calculation of the two-point functions on the cylinder. The main problem is to compute the one-point functions of $\mathbf{l}_{-N}\bar{\mathbf{l}}_{-N}\Phi_{\alpha+2\frac{1-\nu}{\nu}}(0)$. We follow the idea of [3] in order to solve this problem. Namely, we use the fermionic basis introduced for CFT in [1] following the study of lattice models [4, 5, 6]. Let us explain this point.

The basic idea of [4, 5, 6, 1] is to consider operators acting on the space of operators, rather than those acting on the space of states. The Virasoro algebra serves as a tool for labeling local fields in the perturbed theory, even though the conformal invariance is broken. So the space of descendants of the exponential field $\Phi_\alpha(0)$ in the sG model is identified as a linear space with the tensor product of Verma modules $\mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha$. With this identification, the operators acting on the sG local fields can be represented as ones acting on Verma modules. Let us give a simplest example.

Regard the compact direction on the cylinder as time. It is well known that the sG model possesses infinitely many local integrals of motion I_{2k-1}, \bar{I}_{2k-1} , which include in particular the Hamiltonian $H = I_1 + \bar{I}_1$ and the momentum $P = I_1 - \bar{I}_1$. They act on local operators by commutators. We denote this action by $\mathbf{i}_{2k-1}, \bar{\mathbf{i}}_{2k-1}$. In our euclidean approach, the commutator is represented as the difference of local densities integrated over $\mathbb{R} + i0$ and $\mathbb{R} - i0$. It is intuitively clear that the one-point functions of the field obtained by this action always vanish: by moving the contours along the compact direction the integrals cancel with each other. To make it work the boundary conditions at $z = \pm\infty$ are chosen appropriately (see Section 2 for the relevant discussion). From the very definition of Zamolodchikov's construction [16], the operators $\mathbf{i}_{2k-1}, \bar{\mathbf{i}}_{2k-1}$ act on $\mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha$ as

$$\mathbf{i}_1 = \mathbf{l}_{-1}, \quad \mathbf{i}_3 = \sum_{n=0}^{\infty} \mathbf{l}_{-n-2} \mathbf{l}_{n-1}, \quad \text{etc.},$$

and similarly for the second chirality. For our goal, however, this nice construction is useless since the one-point functions of these descendants vanish. It means that the one-point function is a linear functional defined on the quotient space

$$\mathcal{V}_\alpha^{\text{quo}} = \mathcal{V}_\alpha / \sum_{k=1}^{\infty} \mathbf{i}_{2k-1} \mathcal{V}_\alpha, \quad \bar{\mathcal{V}}_\alpha^{\text{quo}} = \bar{\mathcal{V}}_\alpha / \sum_{k=1}^{\infty} \bar{\mathbf{i}}_{2k-1} \bar{\mathcal{V}}_\alpha.$$

Notice that logically this procedure is the same as considering the cohomologies of the affine Jacobi variety in the classical case [7, 8].

Now we proceed to a construction more interesting to us. In the paper [6] it was shown, for the six-vertex model on the cylinder, that the expectation values of quasi-local operators can be simplified by considering certain fermions acting on them. As explained in [3], this construction can be easily generalised to an inhomogeneous six-vertex model. Considering the CFT and the sG models as scaling limits of homogeneous and inhomogeneous six-vertex models, respectively, we conclude that the action of our fermions allows the same identification as the action of the local

integrals of motion. Let us be more specific. We have two kinds of fermions β_{2j-1}^* and γ_{2j-1}^* . The indices agree with the scaling dimension in CFT. The bilinear combinations $\beta_{2j-1}^* \gamma_{2j-1}^*$ act from \mathcal{V}_α to itself, and similarly for the second chirality. Altogether the quotient space $\mathcal{V}_\alpha^{\text{quo}} \otimes \bar{\mathcal{V}}_\alpha^{\text{quo}}$ allows a fermionic basis:

$$(1.4) \quad \beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha(0),$$

where $\#(I^+) = \#(I^-)$, $\#(\bar{I}^+) = \#(\bar{I}^-)$. We set generally

$$(1.5) \quad \begin{aligned} I^+ &= \{2i_1^+ - 1, \dots, 2i_p^+ - 1\} \quad (1 \leq i_1^+ < \dots < i_p^+), & \beta_{I^+}^* &= \beta_{2i_1^+ - 1}^* \cdots \beta_{2i_p^+ - 1}^*, \\ I^- &= \{2i_1^- - 1, \dots, 2i_q^- - 1\} \quad (1 \leq i_1^- < \dots < i_q^-), & \gamma_{I^-}^* &= \gamma_{2i_1^- - 1}^* \cdots \gamma_{2i_q^- - 1}^*, \end{aligned}$$

and similarly for the second chirality.

The quotient space $\mathcal{V}_\alpha^{\text{quo}} \otimes \bar{\mathcal{V}}_\alpha^{\text{quo}}$ can also be realised as a module created from $\Phi_\alpha(0)$ by the action of \mathbf{l}_{-2k} , $\bar{\mathbf{l}}_{-2k}$. The fermionic basis (1.4) can be related to this basis of $\mathcal{V}_\alpha^{\text{quo}} \otimes \bar{\mathcal{V}}_\alpha^{\text{quo}}$ as is explained in [1]. On the other hand, for the operators (1.4) the one-point functions can be computed. This has been done in [3] for the sG model on the plane. We shall show that the case of a cylinder can be dealt with by a simple generalisation.

However, there is an important difference from the paper [3]. There we did not need to compute the one-point functions of the primary fields because they have been known [9]. Now the situation is different. These one-point functions are unknown on the cylinder, but they are needed for the application to OPE. The main new result of the present paper is that we found a fermionic description for them.

First, let us correct one mistake done in the paper [1]. There we seriously considered only the quadratic expressions $\beta_{2j-1}^* \gamma_{2k-1}^*$. For the individual operator β_{2j-1}^* it was erroneously stated that it acts from \mathcal{V}_α to $\mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}}$. Obviously, this is impossible for dimensional reasons. As usual in CFT one has to introduce the screening operators in order to correct this. Surprisingly enough, these screening operators can be constructed from the same material as γ_{2j-1}^* . Let us be more precise. The operators γ_{2j-1}^* are obtained as coefficients of the asymptotical expansion at $\lambda = \infty$ of a generating function $\gamma^*(\lambda)$, where the asymptotics goes in the powers $\lambda^{-\frac{2j-1}{\nu}}$. In the weak sense the function $\gamma^*(\lambda)$ is analytical, and we can define its expansion at $\lambda = 0$ which goes in powers $\lambda^{-\alpha+2j}$. The corresponding coefficients are denoted $\gamma_{\text{screen},j}^*$. Similarly the operators $\bar{\beta}_{2j-1}^*$ are the coefficients of the asymptotics of $\bar{\beta}^*(\lambda)$ at $\lambda = 0$, and we introduce $\bar{\beta}_{\text{screen},j}^*$ as coefficients of its power series at ∞ (the series goes in $\lambda^{\alpha-2j}$).

Let us introduce some notation. We use the multi-index

$$I(m) = \{1, 2, \dots, m\},$$

and define

$$(1.6) \quad \gamma_{\text{screen},I(m)}^* = \gamma_{\text{screen},m}^* \cdots \gamma_{\text{screen},1}^*, \quad \bar{\beta}_{\text{screen},I(m)}^* = \bar{\beta}_{\text{screen},1}^* \cdots \bar{\beta}_{\text{screen},m}^*.$$

Then the m -fold screened primary field is by definition

$$(1.7) \quad \Phi_\alpha^{(m)}(0) = i^m \mu^{2m} \prod_{j=1}^m \cot \frac{\pi\nu}{2}(2j - \alpha) \times \bar{\beta}_{\text{screen}, I(m)}^* \gamma_{\text{screen}, I(m)}^* \Phi_\alpha(0),$$

where the multiplier in the right hand side is introduced for future convenience.

We claim that the basis of the quotient space $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}} \otimes \bar{\mathcal{V}}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$ for $m > 0$ can be constructed as

$$(1.8) \quad \beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha^{(m)}(0),$$

where $\#(I^+) = \#(I^-) + m$, $\#(\bar{I}^-) = \#(\bar{I}^+) + m$.

This is not an abstract statement, but the identification is done by explicit formulae. In particular, taking the element of the lowest dimension we obtain the primary field:

$$(1.9) \quad \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \cong C_m(\alpha) \beta_{I_{\text{odd}}(m)}^* \bar{\gamma}_{\bar{I}_{\text{odd}}(m)}^* \Phi_\alpha^{(m)}(0),$$

where $C_m(\alpha)$ is an important constant (see (6.11) below), and

$$I_{\text{odd}}(m) = 2I(m) - 1 = \{1, 3, \dots, 2m - 1\}.$$

From now on we use the following notation for multi-indices

$$(1.10) \quad \text{if } J = \{j_1, \dots, j_p\} \text{ then } aJ + b = \{aj_1 + b, \dots, aj_p + b\}.$$

The operators on the two sides of (1.9) belong *a priori* to different spaces. By the symbol \cong we imply that the two vectors from different spaces act as the same local operator in CFT. This statement is not a mathematical theorem, because we cannot check completely the identification. What we can compute are the three-point functions with two primary fields of the same scaling dimension $\Delta_{\kappa+1}$. The main result of our fermionic construction is that for the right hand side of (1.9) this three-point function can be evaluated in terms of one function $\omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa, \alpha)$. We computed the asymptotics of this function for $\kappa \rightarrow \infty$ up to κ^{-8} . In the left hand side we use for the three-point function Dotsenko-Fateev formula [19]. The comparison of the asymptotics goes in an amazingly nice way. There are certain consistency conditions which we also check. We compute also the constant $C_m(\alpha)$ and explain its relation to the Lukyanov-Zamolodchikov one-point function.

Consider the part of the sum in the right hand side of (1.3) corresponding to descendants of the primary field $\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)$, $m \geq 0$. Modulo the descendants generated by the local integrals of motion it can be rewritten in the fermionic basis:

$$(1.11) \quad (\mu^2 r^{2\nu})^m r^{2m^2(1-\nu)+2\alpha m\nu} \sum_{N, \bar{N}} C_{\alpha_1, \alpha_2}^{m, N, \bar{N}} (\mu^4 r^{4\nu}) z^{|N|} \bar{z}^{|\bar{N}|} \mathbf{1}_{-N} \bar{\mathbf{1}}_{-\bar{N}} \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \\ \cong (\mu^2 r^{2\nu})^m r^{-2\nu m^2 + 2\alpha m\nu} \sum_{I, J, \bar{I}, \bar{J}} \tilde{C}_{\alpha_1, \alpha_2}^{I^+, I^-, \bar{I}^+, \bar{I}^-} (\mu^4 r^{4\nu}) \\ \times z^{|I^+|+|I^-|} \bar{z}^{|\bar{I}^+|+|\bar{I}^-|} \beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha^{(m)}(0).$$

The structure functions $\tilde{C}_{\alpha_1, \alpha_2}^{I, J, \bar{I}, \bar{J}}(t)$ can be in principle recalculated from $C_{\alpha_1, \alpha_2}^{m, N, \bar{N}}(t)$ because the relation between the usual basis of Verma module and the fermionic basis can be obtained [1]. Nevertheless, we believe that it should be possible to find a direct way to compute $\tilde{C}_{\alpha_1, \alpha_2}^{I^+, I^-, \bar{I}^+, \bar{I}^-}(t)$. For, the elements of the fermionic basis have very simple one-point functions, and hence the structure functions associated with them should also have a fundamental meaning. Let us now turn to the discussion of the one-point functions.

The one-point functions of $\beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha^{(m)}(0)$ are given by determinant formulae. We introduce the function $\omega_R^{\text{sG}}(\lambda, \mu|\alpha)$ by a TBA-like equation written in Section 7. Equation of this kind was written for the first time in [10].

It is very convenient to use instead of $\omega_R^{\text{sG}}(\lambda, \mu|\alpha)$ the function $\Theta_R^{\text{sG}}(l, j|\alpha)$ which is related to the Mellin transform of $\omega_R^{\text{sG}}(\lambda, \mu|\alpha)$ as in (7.8). Then our main formula reads

$$(1.12) \quad \frac{\langle \beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha^{(m)}(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} = \mu^{2m\alpha - 2m^2 + \frac{1}{\nu}(|I^+| + |I^-| + |\bar{I}^+| + |\bar{I}^-|)} \mathcal{D}_R^{\text{sG}}(I^+ \cup (-\bar{I}^+) \mid I^- \cup (-\bar{I}^-)|\alpha),$$

where for $A = \{a_j\}_{j=1, \dots, n}$, $B = \{b_j\}_{j=1, \dots, n}$ we set

$$\begin{aligned} \mathcal{D}_R^{\text{sG}}(A|B|\alpha) &= \prod_{j=1}^n \text{sgn}(a_j) \text{sgn}(b_j) \\ &\times \left(\frac{i}{2\pi\nu^2} \right)^n \det \left(\Theta_R^{\text{sG}} \left(\frac{ia_j}{2\nu}, \frac{ib_k}{2\nu} | \alpha \right) - \text{sgn}(a_j) \delta_{a_j, -b_k} 2\pi\nu \cot \frac{\pi}{2\nu} (a_j + \nu\alpha) \right) \Big|_{j,k=1, \dots, n}. \end{aligned}$$

The mass of soliton M is related to μ by the famous formula [11]:

$$(1.13) \quad \mu = \left[M \frac{\sqrt{\pi} \Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})} \Gamma(\nu)^{-\frac{1}{\nu}} \right]^\nu.$$

Notice that in this paper we do not use this formula as an input, but it comes as a result of our computations. The operators $\gamma_{\text{screen}, j}^*$, $\bar{\beta}_{\text{screen}, j}^*$ are very important in the CFT computations. However, their contribution to the formula (1.12) enters as a decoupled diagonal block in the determinant, and cancels with the multiplier in the definition of $\Phi_\alpha^{(m)}(0)$ (1.7) which is introduced exactly for this reason.

The formula (1.12) requires certain consistency relations for the function $\Theta_R^{\text{sG}}(l, j|\alpha)$ because one can obtain the primary field $\Phi_{\alpha+2m\frac{1-\nu}{\nu}}$ either directly from Φ_α or in m steps passing through $\Phi_{\alpha+2k\frac{1-\nu}{\nu}}$ ($k = 1, \dots, m-1$). One can also write a formula (see (6.12)) for the descendants similar to (1.9) for the primary field; it requires yet another consistency relation. We write these relations down and prove them in Section 9.

From the computations of Section 9 we draw one more conclusion. Until now we considered only the descendants of $\Phi_{\alpha+2m\frac{1-\nu}{\nu}}$ for $m \geq 0$. The question is what to do with $\Phi_{\alpha+2m\frac{1-\nu}{\nu}}$ for $m < 0$ which are also present in (1.3). The direct way to tackle this problem consists in considering the domain $0 > \alpha > -2$ and analytically

continuing from there. But actually this is not needed. The consistency relations imply the existence of the screened primary field $\Phi_\alpha^{(m)}$ for $m < 0$ such that

$$\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \cong C_m(\alpha)\beta_{I_{\text{odd}}(m)}^*\bar{\gamma}_{I_{\text{odd}}(m)}^*\Phi_\alpha^{(m)}(0),$$

where $I_{\text{odd}}(m) = -I_{\text{odd}}(-m)$, and the constant $C_m(\alpha)$ for negative m is given in Section 9 (see (9.10)). A basis of the corresponding Verma module is constructed as

$$(1.14) \quad \beta_{I^+}^*\bar{\beta}_{\bar{I}^+}^*\bar{\gamma}_{\bar{I}^-}^*\gamma_{I^-}^*\Phi_\alpha^{(m)}(0),$$

where $\#(I^+) = \#(I^-) + m$, $\#(\bar{I}^-) = \#(\bar{I}^+) + m$. The difference from the previous case is that the number of β^* 's ($\bar{\gamma}^*$'s) is smaller than the number of γ^* 's ($\bar{\beta}^*$'s). It is useful to keep in mind the analogy with the Dirac fermions with different vacua. Then the parameter α which counts the primary fields is nothing but the value of the zero-mode. This reminds us very much of the construction in the paper [8].

The plan of the paper is as follows. In Section 2 we discuss the scaling limit of the XXZ model in the homogeneous and the inhomogeneous cases which lead respectively to CFT and the sG model. Section 3 deals with the fermionic construction for chiral CFT. Here we introduce the screening fermions. In section 4 we present the fermionic construction of primary fields. We explain the construction of descendants in Section 5. In Section 6 we compare the three-point functions constructed through fermions with the usual CFT formulae. In section 7 we continue the fermionic description of the sG model and present the pairings of fermions. The main formula for the one-point functions is given in Section 8. Section 9 deals with the crucial issue of consistency. In Section 10 we show agreement of our results with several known formulae. Some concluding remarks are given in Section 10. In Appendix A we provide necessary formulae concerning the three-point functions in CFT. In Appendix B several terms of asymptotic expansion of the function $\omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa, \alpha)$ are presented.

2. SCALING LIMIT IN HOMOGENEOUS AND INHOMOGENEOUS CASES

Our study of continuous models is based on the scaling limit of the homogeneous or the inhomogeneous six vertex models on the cylinder. In this section we repeat several definitions and correct some inaccuracies committed in [1, 3]. We start with the infinite tensor product of \mathbb{C}^2 which is denoted by $\mathfrak{H}_{\mathbf{S}}$. In the inhomogeneous case we have alternating parameters $\zeta_0^{\pm 2}$ attached to every component of the tensor product. We define the operator $S(0) = \frac{1}{2} \sum_{j=-\infty}^0 \sigma_j^3$. Then the vector space $\mathcal{W}_{\alpha-s,s}$ consists of the operators $q^{2(\alpha-s)S(0)}\mathcal{O}^{(s)}$ with $\mathcal{O}^{(s)}$ being local and of spin s . All the operators which we consider later act on the space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}.$$

In [5] we have defined the creation operators $\mathbf{t}^*(\zeta)$, $\mathbf{b}_{\text{rat}}^*(\zeta)$, $\mathbf{c}_{\text{rat}}^*(\zeta)$ and the annihilation operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$, all acting on $\mathcal{W}^{(\alpha)}$. We follow the notation of [1]. Actually, the definition of these operators is different in the homogeneous and the inhomogeneous cases, but using formulae from [5] one can easily figure out what they

are. We do not make distinction between the homogeneous and the inhomogeneous cases notationally, all the explanations will be given at proper places.

The creation operators are not defined uniquely. One can apply Bogolubov transformations as we shall discuss soon. Since we shall have a variety of different possibilities, in which one can be easily lost, it is important to start from the safe ground provided by the operators $\zeta^{-\alpha} \mathbf{b}_{\text{rat}}^*(\zeta)$, $\zeta^\alpha \mathbf{c}_{\text{rat}}^*(\zeta)$. They are defined uniquely by the requirement:

$$\text{Tr}_S(\mathbf{b}_{\text{rat}}^*(\zeta)(X)) = 0, \quad \text{Tr}_S(\mathbf{c}_{\text{rat}}^*(\zeta)(X)) = 0, \quad \forall X \in \mathcal{W}^{(\alpha)}.$$

Another important property is that $\zeta^{-\alpha} \mathbf{b}_{\text{rat}}^*(\zeta)(X)$ and $\zeta^\alpha \mathbf{c}_{\text{rat}}^*(\zeta)(X)$ are rational functions of ζ^2 .

The operator $\mathbf{t}^*(\zeta)$ lies in the centre, so, we shall handle it as a \mathbb{C} -number. Actually, for different reasons we shall aim at having $\mathbf{t}^*(\zeta) = 2$ in our final formulae. This explains that the formulae in [3] were written in the quotient space $\mathcal{W}^{(\alpha)}/(\mathbf{t}^*(\zeta) - 2)\mathcal{W}^{(\alpha)}$. However, it is not quite consistent to work in the quotient space from the very beginning, so, for the time being we keep $\mathbf{t}^*(\zeta)$.

Now following [6, 1] we define the operators $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ by the Bogolubov transformation.

$$(2.1) \quad \mathbf{b}^*(\zeta) = e^{-\tilde{\Omega}} \mathbf{b}_{\text{rat}}^*(\zeta) e^{\tilde{\Omega}}, \quad \mathbf{c}^*(\zeta) = e^{-\tilde{\Omega}} \mathbf{c}_{\text{rat}}^*(\zeta) e^{\tilde{\Omega}},$$

where

$$(2.2) \quad \tilde{\Omega} = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma} \tilde{\omega}(\zeta, \xi | \alpha) \mathbf{c}(\xi) \mathbf{b}(\zeta) \frac{d\zeta^2}{\zeta^2} \frac{d\xi^2}{\xi^2},$$

and

$$\tilde{\omega}(\zeta, \xi | \alpha) = - [\Delta_\zeta + 2\mathbf{t}^*(\xi) - 2\mathbf{t}^*(\zeta) + 4\delta_\zeta^- \delta_\xi^- \Delta_\zeta^{-1}] \psi(\zeta/\xi, \alpha),$$

we denote by δ_ζ^- and Δ_ζ the following finite difference operators

$$\begin{aligned} \delta_\zeta^- f(\zeta) &= f(\zeta q) - \frac{1}{2} \mathbf{t}^*(\zeta) f(\zeta), \\ \Delta_\zeta f(\zeta) &= f(\zeta q) - f(\zeta q^{-1}). \end{aligned}$$

We set

$$\psi(\zeta) = \zeta^\alpha \frac{\zeta^2 + 1}{2(\zeta^2 - 1)}.$$

We do not explain how the transcendental function $\delta_\zeta^- \delta_\xi^- \Delta_\zeta^{-1} \psi(\zeta/\xi, \alpha)$ is understood (see [1]). Later we shall write an explicit formula in a particular case needed in this paper. The importance of the operators $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ clearly follows from [6, 1], we shall comment on it later.

Using the formulae of the subsection 2.5 in [5] one concludes that

$$(2.3) \quad \zeta^{-\alpha} \mathbf{b}^*(\zeta) \rightarrow 0, \quad \zeta^\alpha \mathbf{c}^*(\zeta) \rightarrow 0, \quad \zeta \rightarrow 0.$$

However, this property does not hold for $\mathbf{b}_{\text{rat}}^*(\zeta)$, $\mathbf{c}_{\text{rat}}^*(\zeta)$ because the function $\tilde{\omega}(\zeta, \xi | \alpha)$ spoils it. It is impossible to define operators which are rational and satisfy (2.3).

We shall define the operator $\mathbf{b}_0^*(\zeta)$ which satisfies “one half” of (2.3), i.e., the latter half only,

$$(2.4) \quad \mathbf{b}_0^*(\zeta) = e^{-\nabla\Omega} \mathbf{b}_{\text{rat}}^*(\zeta) e^{\nabla\Omega}, \quad \mathbf{c}_0^*(\zeta) = e^{-\nabla\Omega} \mathbf{c}_{\text{rat}}^*(\zeta) e^{\nabla\Omega},$$

where the operator $\nabla\Omega$ is constructed as in (2.2) replacing $\tilde{\omega}(\zeta, \xi|\alpha)$ by

$$\begin{aligned} \nabla\omega(\zeta, \xi|\alpha) &= [2\mathbf{t}^*(\zeta) - 2\mathbf{t}^*(\xi) - \Delta_\zeta] \psi(\zeta/\xi, \alpha) \\ &\quad + \frac{(2q^\alpha - \mathbf{t}^*(\zeta))(2q^{-\alpha} - \mathbf{t}^*(\xi))}{2(q^\alpha - q^{-\alpha})} \left(\frac{\zeta}{\xi}\right)^\alpha. \end{aligned}$$

Combining (2.1) with (2.4) we obtain

$$(2.5) \quad \mathbf{b}^*(\zeta) = e^{-\Omega_0} \mathbf{b}_0^*(\zeta) e^{\Omega_0}, \quad \mathbf{c}^*(\zeta) = e^{-\Omega_0} \mathbf{c}_0^*(\zeta) e^{\Omega_0},$$

where Ω_0 given by (2.2) with $\tilde{\omega}(\zeta, \xi|\alpha)$ replaced by $4\omega_0(\zeta, \xi|\alpha)$, and

$$(2.6) \quad \omega_0(\zeta, \xi|\alpha) = -\delta_\zeta^- \delta_\xi^- \Delta_\zeta^{-1} \psi_0(\zeta/\xi, \alpha), \quad \psi_0(\zeta, \alpha) = \frac{\zeta^\alpha}{\zeta^2 - 1}.$$

We note that

$$\tilde{\omega}(\zeta, \xi|\alpha) = \nabla\omega(\zeta, \xi|\alpha) + 4\omega_0(\zeta, \xi|\alpha).$$

Now it is easy to see that since we are working in

$$0 < \alpha < 2,$$

the following estimates hold,

$$\mathbf{b}_0^*(\zeta) = O(\zeta^\alpha), \quad \mathbf{c}_0^*(\zeta) = O(\zeta^{2-\alpha}), \quad \zeta \rightarrow 0.$$

The analytical structure of the creation operators depends on the target space. If the latter is $\mathcal{W}_{\alpha,0}$ we have for $\mathbf{b}_0^*(\zeta)$, $\mathbf{c}_0^*(\zeta)$:

$$(2.7) \quad \mathbf{b}_0^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{\alpha-2+2j} \mathbf{b}_{\text{screen},j}^*, \quad \mathbf{c}_0^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{-\alpha+2j} \mathbf{c}_{\text{screen},j}^*,$$

the meaning of the suffix “screen” will be clear from what follows.

Now let us return to the main problem solved in [6]. We start with the six-vertex model on an infinite cylinder. We call the infinite direction the “space direction”, and the compact direction the “Matsubara direction”. We associate with them the spaces $\mathfrak{H}_{\mathbf{S}}$, $\mathfrak{H}_{\mathbf{M}}$. The number of sites in the Matsubara direction is even, and is denoted by \mathbf{n} . In the present paper we shall consider two cases: homogeneous six vertex model which gives the chiral CFT in the scaling limit [1], and inhomogeneous six vertex model which gives the sG model in the scaling limit. In both cases we denote by $T_{\mathbf{S},\mathbf{M}}$ the rectangular monodromy matrix representing the universal R -matrix in the tensor product $\mathfrak{H}_{\mathbf{S}} \otimes \mathfrak{H}_{\mathbf{M}}$:

$$(2.8) \quad T_{j,\mathbf{M}} = \prod_{j=-\infty}^{\infty} T_{j,\mathbf{M}}(1), \quad T_{j,\mathbf{M}}(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} L_{j,\mathbf{m}}(\zeta), \quad \text{homogeneous,}$$

$$(2.9) \quad T_{\mathbf{S},\mathbf{M}} = \prod_{j=-\infty}^{\infty} T_{j,\mathbf{M}}(\zeta_0^{(-1)^j}), \quad T_{j,\mathbf{M}}(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} L_{j,\mathbf{m}}(\zeta \zeta_0^{-(-1)^{\mathbf{m}}}), \quad \text{inhomogeneous.}$$

In the paper [6] we considered the functional

$$(2.10) \quad Z_{\mathbf{n}}^{\kappa} \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\text{Tr}_S \text{Tr}_M \left(T_{S,M} q^{2\kappa S + 2\alpha S(0)} \mathcal{O} \right)}{\text{Tr}_S \text{Tr}_M \left(T_{S,M} q^{2\kappa S + 2\alpha S(0)} \right)}.$$

The main theorem of [6] says that

$$(2.11) \quad Z_{\mathbf{n}}^{\kappa} \{ \mathbf{t}^*(\zeta)(X) \} = 2\rho_{\mathbf{n}}(\zeta|\kappa, \kappa + \alpha) Z_{\mathbf{n}}^{\kappa} \{ X \},$$

$$(2.12) \quad Z_{\mathbf{n}}^{\kappa} \{ \mathbf{b}^*(\zeta)(X) \} = \frac{1}{2\pi i} \oint_{\Gamma} \boldsymbol{\omega}_{\mathbf{n}}(\zeta, \xi|\kappa, \alpha) Z_{\mathbf{n}}^{\kappa} \{ \mathbf{c}(\xi)(X) \} \frac{d\xi^2}{\xi^2},$$

$$(2.13) \quad Z_{\mathbf{n}}^{\kappa} \{ \mathbf{c}^*(\zeta)(X) \} = -\frac{1}{2\pi i} \oint_{\Gamma} \boldsymbol{\omega}_{\mathbf{n}}(\xi, \zeta|\kappa, \alpha) Z_{\mathbf{n}}^{\kappa} \{ \mathbf{b}(\xi)(X) \} \frac{d\xi^2}{\xi^2},$$

where the integration contour goes around 1 for the homogeneous case and around $\zeta_0^{\pm 2}$ for the inhomogeneous one. We use the boldface letter for $\boldsymbol{\omega}_{\mathbf{n}}(\xi, \zeta|\kappa, \alpha)$ in order to distinguish it from several auxiliary ω 's which we had before.

A distinguished feature of $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ is that in the simple-minded limit $\mathbf{n} \rightarrow \infty$ we have

$$Z_{\infty}^{\kappa} \{ \mathbf{b}^*(\zeta)(X) \} = 0, \quad Z_{\infty}^{\kappa} \{ \mathbf{c}^*(\zeta)(X) \} = 0.$$

The word “simple-minded” means that \mathbf{n} goes to ∞ without rescaling the Bethe roots for the Matsubara transfer-matrix.

Certainly, the same kind of relations is true if we put \mathbf{b}_0^* and \mathbf{c}_0^* in the left hand sides, and replace $\boldsymbol{\omega}_{\mathbf{n}}$ by $\boldsymbol{\omega}_{\mathbf{n}} + 4\omega_0$ in the right hand sides. We have

$$\rho_{\mathbf{n}}(\zeta|\kappa, \kappa + \alpha) = \frac{T_{\mathbf{n}}(\zeta|\alpha + \kappa)}{T_{\mathbf{n}}(\zeta|\kappa)},$$

where $T_{\mathbf{n}}(\zeta|\alpha + \kappa)$, $T_{\mathbf{n}}(\zeta|\kappa)$ are maximal eigenvalues of the twisted Matsubara transfer-matrices [6]. We do not write the equation for $\boldsymbol{\omega}_{\mathbf{n}}(\zeta, \xi|\kappa, \alpha)$ here. It can be found in [6, 12, 1].

Following [1] we introduce a generalised functional $Z_{\mathbf{n}}^{\kappa, -s}$, then we shall comment on its relevance to the scaling limit in the homogeneous and the inhomogeneous cases. If the target of the operator $\zeta^{\alpha} \mathbf{c}_0^*(\zeta)$ is $\mathcal{W}_{\alpha+s, -s}$ it develops a singularity at $\zeta^2 = 0$, namely,

$$(2.14) \quad \mathbf{c}_0^*(\zeta) = \sum_{j=0}^{s-1} \zeta^{-\alpha-2j} \mathbf{c}_{\text{screen}, -j}^* + \mathbf{c}_{0, \text{reg}}^*(\zeta), \quad \mathbf{c}_{0, \text{reg}}^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{-\alpha+2j} \mathbf{c}_{\text{screen}, j}^*.$$

Using the singular part of $\zeta^{\alpha+2} \mathbf{c}_0^*(\zeta)$ we define:

$$(2.15) \quad Z_{\mathbf{n}}^{\kappa, -s} \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\text{Tr}_S \text{Tr}_M \left(Y_{\mathbf{M}}^{(s)} T_{S,M} q^{2\kappa S} \mathbf{c}_{\text{screen}, -0}^* \cdots \mathbf{c}_{\text{screen}, -s+1}^* (q^{2\alpha S(0)} \mathcal{O}) \right)}{\text{Tr}_S \text{Tr}_M \left(Y_{\mathbf{M}}^{(s)} T_{S,M} q^{2\kappa S} \mathbf{c}_{\text{screen}, -0}^* \cdots \mathbf{c}_{\text{screen}, -s+1}^* (q^{2\alpha S(0)}) \right)},$$

where $Y_{\mathbf{M}}^{(s)}$ is an operator of spin s acting only in $\mathfrak{H}_{\mathbf{M}}$. This functional possesses several nice properties. First, it is independent of $Y_{\mathbf{M}}^{(s)}$ if the latter is in general position [1]. Second, formulae (2.11), (2.12), (2.13) remain valid provided the functions $\rho_{\mathbf{n}}(\zeta|\kappa + \alpha, \kappa)$ and $\omega_{\mathbf{n}}(\zeta, \xi|\kappa, \alpha)$ are replaced by appropriate counterparts. In particular,

$$(2.16) \quad \rho_{\mathbf{n}}(\zeta|\kappa, \alpha, -s) = \frac{T_{\mathbf{n}}(\zeta|\alpha + \kappa + s, -s)}{T_{\mathbf{n}}(\zeta|\kappa)},$$

where $T_{\mathbf{n}}(\zeta|\alpha + \kappa + s, -s)$ is the maximal eigenvalue of the Matsubara transfer-matrix with twist $\alpha + \kappa + s$ in the space of spin $-s$. For later reference we record also

$$(2.17) \quad Z_{\mathbf{n}}^{\kappa, -s} \left\{ \mathbf{b}^*(\zeta) \mathbf{c}^*(\xi) (q^{2\alpha S(0)}) \right\} = \omega_{\mathbf{n}}(\zeta, \xi|\kappa, \alpha, -s),$$

$$(2.18) \quad Z_{\mathbf{n}}^{\kappa, -s} \left\{ \mathbf{b}_0^*(\zeta) \mathbf{c}_0^*(\xi) (q^{2\alpha S(0)}) \right\} = \omega_{\mathbf{n}}(\zeta, \xi|\kappa, \alpha, -s) + 4\omega_0(\zeta, \xi|\alpha).$$

Now we shall discuss the importance of the definition (2.15) for the scaling limit in the homogeneous and the inhomogeneous cases.

Homogeneous case. Recall [1] that introducing the step of the lattice a we consider for the homogeneous chain the scaling limit

$$(2.19) \quad \mathbf{n} \rightarrow \infty, \quad a \rightarrow 0, \quad \mathbf{n}a = 2\pi \text{ fixed}.$$

Actually in [1] we wrote $2\pi R$ in the right hand side of the last formula, but in this paper we prefer to set $R = 1$ in order to avoid confusion with the sG case. Anyway, in the conformal case the dependance on the radius can be easily reconstituted. The spectral parameter ζ is subject to rescaling

$$(2.20) \quad \zeta = \lambda(Ca)^\nu,$$

where λ is finite and

$$C = \frac{\Gamma\left(\frac{1-\nu}{2\nu}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{2\nu}\right)} \Gamma(\nu)^{\frac{1}{\nu}}.$$

We have

$$\lim_{a \rightarrow 0} \rho_{\mathbf{n}}(\zeta|\kappa, \alpha, -s) = \rho(\lambda|\kappa, \kappa') = \frac{T^{\text{sc}}(\lambda|\kappa')}{T^{\text{sc}}(\lambda|\kappa)}, \quad \kappa' = \kappa + \alpha - 2s \frac{1-\nu}{\nu},$$

where $T^{\text{sc}}(\lambda|\kappa)$ is the maximal eigenvalue of the BLZ transfer-matrix [13, 14]. By analytical continuation we allow κ' to be arbitrary.

The main statement of [1] is that the functional $Z^{\kappa, -s}$ scales to the normalised three point function in CFT. Namely if a quasi-local operator $q^{2\alpha S(0)} \mathcal{O}$ tends to a Virasoro descendant $P_{\alpha, \mathcal{O}}(\{1_{-k}\}) \Phi_{\alpha}(0)$ with $P_{\alpha, \mathcal{O}}$ a polynomial, then

$$(2.21) \quad \lim_{a \rightarrow 0} Z^{\kappa, -s} \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\langle 1 - \kappa' | P_{\alpha, \mathcal{O}}(\{1_{-k}\}) \Phi_{\alpha}(0) | 1 + \kappa \rangle}{\langle 1 - \kappa' | \Phi_{\alpha}(0) | 1 + \kappa \rangle},$$

where the right hand side is a ratio of three-point functions in CFT with central charge

$$c = 1 - 6 \frac{\nu^2}{1 - \nu}.$$

For technical reasons, we have been able to treat quantitatively only the case $\kappa = \kappa'$ [1]. In that case

$$(2.22) \quad \rho(\lambda|\kappa, \kappa) = 1,$$

and we can factor out the action of $\mathbf{t}^*(\zeta)$ since (2.22) implies

$$(2.23) \quad \lim_{a \rightarrow 0} \mathbf{t}^*(\zeta) = 2.$$

So, this important condition appears in CFT case as a technical requirement.

Inhomogeneous case. In this case we follow [15]. The main idea is to introduce again the step of the lattice a and radius R related by

$$\mathbf{n}a = 2\pi R,$$

and to consider the limit

$$a \rightarrow 0, \quad \zeta_0 \rightarrow \infty, \quad \text{keeping } R \text{ and } M = 4a^{-1}\zeta_0^{-\frac{1}{\nu}} \text{ fixed.}$$

Then we are supposed to obtain the Euclidean sG model on a cylinder of radius R with the mass of soliton equal to M . We want to consider

$$\frac{\langle P(\{\mathbf{1}_{-k}\}, \{\bar{\mathbf{1}}_{-k}\}) \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}}.$$

the descendants are uniquely defined by their conformal limit and the absence of finite counterterms which can be guaranteed by dimensional reasons. By dimensional reasons, for irrational ν , the descendants are uniquely defined by their conformal limit and the absence of finite counterterms.

The first naïve idea would be to consider the scaling limit of $Z_{\mathbf{n}}^0$. We set $\kappa = 0$ in order to simplify the formulae, though one can easily generalise. However, this idea immediately proves to be wrong because of our old enemy $\mathbf{t}^*(\zeta)$. Indeed, according to (2.11) it gives a non-trivial contribution to $Z_{\mathbf{n}}^0$. But we know that $\mathbf{t}^*(\zeta)$ is the generating function of the adjoint action of the local integrals of motion. So, nontrivial $\rho_{\mathbf{n}}(\zeta|\alpha, 0)$ breaks this invariance, in particular, it breaks the translational invariance.

What is the reason for that? The point is that due to the bosonisation rule

$$\lim_{a \rightarrow 0} a^{-1} \sigma_j^3 = \frac{1}{i\pi(1-\nu)} \partial_x \varphi(x),$$

we have

$$\lim_{a \rightarrow 0} q^{2\alpha S(0)} = \Phi_{-\alpha}(-\infty) \Phi_\alpha(0).$$

So, the scaling limit of $Z_{\mathbf{n}}\{q^{2\alpha S(0)}\}$ corresponds rather to the two-point function with one field placed at $-\infty$ than to the one-point function. There is no simple way to define an analogue of the field $\Phi_\alpha(0)$ itself on the lattice. So, we have to take an indirect way in order to "screen out" the field $\Phi_{-\alpha}(-\infty)$, and to restore the invariance under the action of the local integrals of motion.

It is true that the difficulty is the same as in the previous case, but the implications are different. While the requirement (2.23) appears as a technical restriction in the

CFT case, it is of direct physical significance in the sG model. We satisfy this requirement in the same way as in CFT assuming that

$$(2.24) \quad \lim_{a \rightarrow 0} Z_{\mathbf{n}}^{-s} \{q^{2\alpha S(0)} \mathcal{O}\} = \frac{\langle P_{\alpha,0}(\{\mathbf{1}_{-k}\}, \{\bar{\mathbf{1}}_{-k}\}) \Phi_{\alpha}(0) \rangle_R^{\text{sG}}}{\langle \Phi_{\alpha}(0) \rangle_R^{\text{sG}}}, \quad \alpha = 2s \frac{1-\nu}{\nu}.$$

Certainly the statements (2.21), (2.24) are rather strong conjectures. We would be happy to have more robust supporting arguments than we have for the moment. Notice that the analytical continuation from the points $\alpha = 2s \frac{1-\nu}{\nu}$ is possible.

3. FERMIONS IN CHIRAL CFT

In this section we study the fermions and their expectation values in chiral CFT. Recall that performing the scaling limit we rescale ζ according to (2.20). In the scaling limit, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ produce operators $\beta^*(\lambda)$, $\gamma^*(\lambda)$ with the asymptotics at $\lambda = \infty$:

$$(3.1) \quad \begin{aligned} \beta^*(\lambda) &= \frac{1}{2} \lim_{a \rightarrow 0} \mathbf{b}^*(\lambda(Ca)^{\nu}), \quad \gamma^*(\lambda) = \frac{1}{2} \lim_{a \rightarrow 0} \mathbf{c}^*(\lambda(Ca)^{\nu}), \\ \beta^*(\lambda) &\simeq \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \beta_{2j-1}^*, \quad \gamma^*(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} \gamma_{2j-1}^*, \end{aligned}$$

while $\mathbf{b}_0^*(\zeta)$, $\mathbf{c}_0^*(\zeta)$ produce operators $\beta_{\text{screen}}^*(\lambda)$, $\gamma_{\text{screen}}^*(\lambda)$ with the asymptotics at $\lambda = 0$:

$$(3.2) \quad \begin{aligned} \beta_{\text{screen}}^*(\lambda) &= \frac{1}{2} \lim_{a \rightarrow 0} \mathbf{b}_0^*(\lambda(Ca)^{\nu}), \quad \gamma_{\text{screen}}^*(\lambda) = \frac{1}{2} \lim_{a \rightarrow 0} \mathbf{c}_0^*(\lambda(Ca)^{\nu}), \\ \beta_{\text{screen}}^*(\lambda) &\simeq \sum_{j=1}^{\infty} \lambda^{\alpha+2j-2} \beta_{\text{screen},j}^*, \quad \gamma_{\text{screen}}^*(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\alpha+2j} \gamma_{\text{screen},j}^*. \end{aligned}$$

We assign scaling dimensions to these operators by the corresponding powers in λ . Namely β_{2j-1}^* , γ_{2j-1}^* carry the scaling dimension $2j-1$, while $\beta_{\text{screen},j}^*$ and $\gamma_{\text{screen},j}^*$ carry the scaling dimensions $\nu(2-\alpha-2j)$ and $\nu(\alpha-2j)$, respectively.

The normalised three-point functions

$$\frac{\langle 1 - \kappa | \beta_{I+}^* \gamma_{I-}^* \beta_{\text{screen},J+}^* \gamma_{\text{screen},J-}^* \phi_{\alpha}(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_{\alpha}(0) | 1 + \kappa \rangle}$$

can be expressed as determinants of pairings. Each pairing is written in terms of a function $\Theta(l, m | \kappa, \alpha)$ introduced in [1]. Its asymptotics for $\kappa \rightarrow \infty$ can be computed by a regular procedure. For reference, in Appendix B we provide the asymptotic expansion up to the order κ^{-6} . Motivated by this asymptotic expansion we conjecture that $\Theta(l, m | \kappa, \alpha) + i/(l+m)$ is an entire function of l, m .

Set

$$(3.3) \quad \begin{aligned} D_{2j-1}(\alpha) &= -\sqrt{\frac{i}{\nu}} \Gamma(\nu)^{-\frac{2j-1}{\nu}} (1-\nu)^{\frac{2j-1}{2}} \frac{\Gamma(\frac{\alpha}{2} + \frac{1}{2\nu}(2j-1))}{(j-1)! \Gamma(\frac{\alpha}{2} + \frac{1-\nu}{2\nu}(2j-1))}, \\ E_j(\alpha) &= \frac{(-1)^{j-1}}{\sqrt{i}} \left(2i \Gamma(\nu)^{\frac{1}{\nu}} \nu^{-1} \right)^{(2j-\alpha)\nu} \frac{\Gamma(\frac{1}{2} + \nu j - \frac{\nu\alpha}{2})}{(j-1)! \Gamma(1 - (1-\nu)j - \frac{\nu\alpha}{2})}. \end{aligned}$$

Notation being as above, we have:

$$(3.4) \quad \frac{\langle 1 - \kappa | \beta_{2j-1}^* \gamma_{2k-1}^* \phi_\alpha(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_\alpha(0) | 1 + \kappa \rangle} = \left(\frac{\nu}{2\sqrt{1-\nu}} \kappa \right)^{2(j+k)-2} \nu^{-1} \\ \times D_{2j-1}(\alpha) D_{2k-1}(2-\alpha) \Theta\left(i \frac{2j-1}{2\nu}, i \frac{2k-1}{2\nu} | \kappa, \alpha\right),$$

$$(3.5) \quad \frac{\langle 1 - \kappa | \beta_{\text{screen},j}^* \gamma_{\text{screen},k}^* \phi_\alpha(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_\alpha(0) | 1 + \kappa \rangle} = \kappa^{-2\nu(j+k-1)} \\ \times E_j(2-\alpha) E_k(\alpha) \Theta\left(-i(j-1+\frac{\alpha}{2}), -i(k-\frac{\alpha}{2}) | \kappa, \alpha\right),$$

$$(3.6) \quad \frac{\langle 1 - \kappa | \beta_{2j-1}^* \gamma_{\text{screen},k}^* \phi_\alpha(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_\alpha(0) | 1 + \kappa \rangle} = \left(\frac{\nu}{2\sqrt{1-\nu}} \kappa \right)^{2j-1} \nu^{-\frac{1}{2}} \kappa^{-2\nu k + \nu\alpha} \\ \times D_{2j-1}(\alpha) E_k(\alpha) \Theta\left(i \frac{2j-1}{2\nu}, -i(k-\frac{\alpha}{2}) | \kappa, \alpha\right),$$

$$(3.7) \quad \frac{\langle 1 - \kappa | \beta_{\text{screen},j}^* \gamma_{2k-1}^* \phi_\alpha(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_\alpha(0) | 1 + \kappa \rangle} = \left(\frac{\nu}{2\sqrt{1-\nu}} \kappa \right)^{2k-1} \nu^{-\frac{1}{2}} \kappa^{-2\nu(j-1)-\nu\alpha} \\ \times E_j(2-\alpha) D_{2k-1}(2-\alpha) \Theta\left(-i(j-1+\frac{\alpha}{2}), i \frac{2k-1}{2\nu} | \kappa, \alpha\right).$$

Let us explain the origin of the formulae (3.4)–(3.7).

Formula (3.4) arises from the scaling limit of (2.17). The function

$$\omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) = \frac{1}{4} \lim_{a \rightarrow 0} \omega_{\mathbf{n}}(\zeta, \xi | \kappa, \alpha, -s)$$

is given by the inverse Mellin transform of the function $\Theta(l, m | \kappa, \alpha)$ [1]:

$$(3.8) \quad \omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) = \frac{1}{2\pi i} \iint dl dm \tilde{S}(l, \alpha) \tilde{S}(m, 2-\alpha) \Theta(l + i0, m | \kappa, \alpha) \\ \times \left(\frac{e^{\frac{\pi i \nu}{2}} \Gamma(\nu) 2^\nu \lambda}{(\nu \kappa)^\nu} \right)^{2il} \left(\frac{e^{\frac{\pi i \nu}{2}} \Gamma(\nu) 2^\nu \mu}{(\nu \kappa)^\nu} \right)^{2im}, \\ \tilde{S}(k, \alpha) = \frac{\Gamma(-ik + \frac{\alpha}{2}) \Gamma(\frac{1}{2} + i\nu k)}{\sqrt{2\pi} \Gamma(-i(1-\nu)k + \frac{\alpha}{2}) (1-\nu)^{\frac{1-\alpha}{2}}}.$$

We obtain (3.4) by closing the contours of integration into the upper half plane and taking the coefficients of powers of λ, μ .

Next consider the scaling limit of (2.18). As was said in Section 2, we consider only the case $\kappa = \kappa'$. So, effectively $\mathbf{t}^*(\zeta) = 2$, and the function $\omega_0(\zeta, \xi | \alpha)$ depends only on the ratio ζ/ξ ,

$$(3.9) \quad \omega_0(\zeta, \xi | \alpha) = \omega_0(\zeta/\xi, \alpha), \\ \omega_0(\lambda, \alpha) = -i \int_{-\infty}^{\infty} \lambda^{2ik} \frac{\sinh \frac{\pi}{2} (2(1-\nu)k + i\alpha)}{2 \sinh \frac{\pi}{2} (2k + i\alpha) \cosh \pi \nu k} dk.$$

We have

$$(3.10) \quad \omega_0(\lambda^{-1}, \alpha) = \omega_0(\lambda, 2-\alpha).$$

Now we have

$$\begin{aligned}
 (3.11) \quad & \omega^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) + \omega_0(\lambda/\mu, \alpha) \\
 &= \frac{1}{2\pi i} \iint dl dm \tilde{S}(l, \alpha) \tilde{S}(m, 2 - \alpha) \Theta(l - i0, m | \kappa, \alpha) \\
 &\quad \times \left(\frac{e^{\frac{\pi i \nu}{2}} \Gamma(\nu) 2^\nu \lambda}{(\nu \kappa)^\nu} \right)^{2il} \left(\frac{e^{\frac{\pi i \nu}{2}} \Gamma(\nu) 2^\nu \mu}{(\nu \kappa)^\nu} \right)^{2im}.
 \end{aligned}$$

Assuming the conjectured analyticity of $\Theta(l, m | \kappa, \alpha)$ and closing the contours into the lower-half plane, we obtain (3.5).

Consider now (3.6). The corresponding generating function is $\beta^*(\lambda) \gamma_{\text{screen}}^*(\mu)$, which seems to pose a problem because we need to close contours into different half-planes. However, the difference between (3.11) and (3.8) is the function $\omega_0(\lambda/\mu, \alpha)$ which for $\lambda \rightarrow \infty$, $\mu \rightarrow 0$ is given by asymptotical series containing $(\lambda/\mu)^{-\frac{2j-1}{\nu}}$ and $(\lambda/\mu)^{\alpha-2j}$ (see (8.1) below). So, this asymptotics does not mix $\lambda^{-\frac{2j-1}{\nu}}$ with $\mu^{-\alpha+2j}$ and we can safely use residues of (3.8) to compute (3.6). Similarly we obtain (3.7).

Of course the entire working carries over to the second chirality, but we have to recalculate ω . Carefully repeating the computations of [1] we get

$$\begin{aligned}
 (3.12) \quad & \bar{\omega}^{\text{sc}}(\lambda, \mu | \kappa, \kappa, \alpha) \simeq \frac{1}{2\pi i} \iint dl dm \tilde{S}(l, 2 - \alpha) \tilde{S}(m, \alpha) \Theta(l + i0, m | -\kappa, 2 - \alpha) \\
 &\quad \times \left(\frac{e^{-\frac{\pi i \nu}{2}} \Gamma(\nu) 2^\nu}{(\nu \kappa)^\nu \lambda} \right)^{2il} \left(\frac{e^{-\frac{\pi i \nu}{2}} \Gamma(\nu) 2^\nu}{(\nu \kappa)^\nu \mu} \right)^{2im}, \\
 & \bar{\omega}_0(\lambda, \mu | \alpha) = \omega_0(\lambda/\mu, 2 - \alpha).
 \end{aligned}$$

These functions are used to compute the pairings for the operators

$$(3.13) \quad \bar{\beta}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{\frac{2j-1}{\nu}} \bar{\beta}_{2j-1}^*, \quad \bar{\gamma}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{\frac{2j-1}{\nu}} \bar{\gamma}_{2j-1}^*,$$

$$(3.14) \quad \bar{\beta}_{\text{screen}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-2j+\alpha} \bar{\beta}_{\text{screen},j}^*, \quad \bar{\gamma}_{\text{screen}}^*(\lambda) = \sum_{j=1}^{\infty} \lambda^{-2j+2-\alpha} \bar{\gamma}_{\text{screen},j}^*.$$

4. FERMIONIC CONSTRUCTION OF PRIMARY FIELDS

Returning to the first chirality, we now explain how to use fermions to construct different primary fields out of one.

Acting on the primary field $\phi_\alpha(0)$, the collection of operators β_{2j-1}^* , γ_{2j-1}^* , $\beta_{\text{screen},j}^*$, $\gamma_{\text{screen},j}^*$ generates a huge space \mathcal{H}_α . Unless stated otherwise, we shall always consider the quotient space modulo the action of the integrals of motion. We know from [1] that the quotient space $\mathcal{V}_\alpha^{\text{quo}}$ is embedded into \mathcal{H}_α with the basis being

$$\beta_{I^+}^* \gamma_{I^-}^* \phi_\alpha(0),$$

where $\#(I^+) = \#(I^-)$. The main statement of the present paper concerning CFT is that the quotient space $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$ can also be embedded into \mathcal{H}_α , with the basis

being

$$(4.1) \quad \beta_{I^+}^* \gamma_{I^-}^* \gamma_{\text{screen}, I(m)}^* \phi_\alpha(0),$$

with $\#(I^+) = \#(I^-) + m$. From the rule for assigning the scaling dimensions, one can easily conclude that the character of the space generated by (4.1) coincides with that of $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$. So, at least our statement makes sense from dimensional point of view.

In particular, the vector of the lowest dimension among (4.1) must be identified with the primary field:

$$(4.2) \quad \phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \cong \beta_{I_{\text{odd}}(m)}^* \gamma_{\text{screen}, I(m)}^* \phi_\alpha(0).$$

It has been said in Introduction that we use the symbol \cong for identifying vectors belonging to different spaces. For example, in the last formula the left hand side belongs to $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$ while the right hand side belongs to \mathcal{H}_α . We identify these vectors on the grounds that their three-point functions on the cylinder coincide, in the presence of primary fields $\phi_{1-\kappa}(-\infty)$, $\phi_{1+\kappa}(\infty)$ for arbitrary κ . Certainly, this is not enough to state a theorem, but this is the best we can do for the moment.

The self-consistency of the formula (4.2) gives rise to an identity among the special values of $\Theta(l, m|\kappa, \alpha)$. There are two ways to construct the field $\phi_{\alpha+4\frac{1-\nu}{\nu}}$ using fermions:

$$\begin{aligned} \phi_{\alpha+4\frac{1-\nu}{\nu}}(0) &\cong \beta_1^* \gamma_{\text{screen}, 1}^* \phi_{\alpha+2\frac{1-\nu}{\nu}}(0) \\ &\cong \beta_1^* \beta_3^* \gamma_{\text{screen}, 2}^* \gamma_{\text{screen}, 1}^* \phi_\alpha(0), \end{aligned}$$

where three fields belong correspondingly to $\mathcal{V}_{\alpha+4\frac{1-\nu}{\nu}}^{\text{quo}}$, $\mathcal{H}_{\alpha+2\frac{1-\nu}{\nu}}$, \mathcal{H}_α . The creation operators $\beta_1^*, \gamma_{\text{screen}, 1}^*$ in the first line and those in the second line are different: the former acting on $\mathcal{H}_{\alpha+2\frac{1-\nu}{\nu}}$, and the latter on \mathcal{H}_α . This compatibility requirement implies the identity

$$\begin{aligned} &\Theta(i/2\nu, -i(1-\alpha/2)|\kappa, \alpha) \Theta(i/2\nu, -i(1-\alpha/2-(1-\nu)/\nu)|\kappa, \alpha+2(1-\nu)/\nu) \\ &= -\frac{1}{4\nu}(\alpha\nu-2\nu+3)(\alpha\nu-4\nu+1) \\ &\times \begin{vmatrix} \Theta(i/2\nu, -i(1-\alpha/2)|\kappa, \alpha) & \Theta(i/2\nu, -i(2-\alpha/2)|\kappa, \alpha) \\ \Theta(3i/2\nu, -i(1-\alpha/2)|\kappa, \alpha) & \Theta(3i/2\nu, -i(2-\alpha/2)|\kappa, \alpha) \end{vmatrix}. \end{aligned}$$

Using the asymptotic expansion of $\Theta(l, m|\kappa, \alpha)$, we have checked this identity up to κ^{-8} . We regard the validity of the identity as another supporting evidence in favour of our statement (4.1). Later in section 9, we shall give a proof of the corresponding identities in the more general setting of the sG model.

5. FERMIONIC CONSTRUCTION OF DESCENDANTS

The descendants of $\phi_\alpha(0)$ can be constructed in the form

$$\beta_{I^+}^* \gamma_{I^-}^* \phi_\alpha(0).$$

They are related to the Virasoro descendants as

$$\begin{aligned} & \beta_{I^+}^* \gamma_{I^-}^* \phi_\alpha(0) \\ &= \prod_{2j-1 \in I^+} D_{2j-1}(\alpha) \prod_{2j-1 \in I^-} D_{2j-1}(2-\alpha) [P_{I^+, I^-}^{\text{even}} + d_\alpha P_{I^+, I^-}^{\text{odd}}] \phi_\alpha(0), \end{aligned}$$

where $D_{2j-1}(\alpha)$ is given in (3.3), and

$$d_\alpha = \frac{\nu(\nu-2)}{\nu-1}(\alpha-1) = \frac{1}{6} \sqrt{(25-c)(24\Delta_\alpha+1-c)}.$$

As it was mentioned already, all formulas are to be understood modulo the action of the integrals of motion. Here $P_{I^+, I^-}^{\text{even}}$, $P_{I^+, I^-}^{\text{odd}}$ are polynomials in the generators $\{\mathbf{l}_{-2k}\}$, whose coefficients depend polynomially on c and rationally on Δ_α . The simplest examples are

$$(5.1) \quad \begin{aligned} P_{\{1\}, \{1\}}^{\text{even}} &= \mathbf{l}_{-2}, \quad P_{\{1\}, \{1\}}^{\text{odd}} = 0, \\ P_{\{1\}, \{3\}}^{\text{even}} &= P_{\{3\}, \{1\}}^{\text{even}} = \mathbf{l}_{-2}^2 + \frac{2c-32}{9} \mathbf{l}_{-4}, \quad P_{\{1\}, \{3\}}^{\text{odd}} = P_{\{3\}, \{1\}}^{\text{odd}} = \frac{2}{3} \mathbf{l}_{-4}. \end{aligned}$$

The descendant $\beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)$ of the shifted primary field belongs to $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$. We want to write another representation for this descendant using our definition of the primary field $\phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)$ in \mathcal{H}_α . Some general considerations and, most importantly, “experimental” data bring us to the following formula:

$$(5.2) \quad \beta_{I^+}^* \gamma_{I^-}^* \phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \cong \beta_{I^++2m}^* \gamma_{I^--2m}^* \beta_{I_{\text{odd}}(m)}^* \gamma_{\text{screen}, I(m)}^* \phi_\alpha(0).$$

There is a trouble here: it is obscure how to understand γ_{-a}^* when the suffix $-a$ becomes negative. A natural idea would be to identify it with the annihilation operator β_a satisfying

$$(5.3) \quad [\beta_a, \beta_b^*]_+ = \delta_{a,b} \varepsilon(a).$$

These operators should originate from $\mathbf{b}(\zeta)$. However, we do not know how to take the continuous limit directly, and normalise these operators. So, we just introduce them by hand, and impose the rule

$$(5.4) \quad \gamma_{b-2m}^* = c(b) \beta_{2m-b} \quad \text{if } b < 2m$$

in the right hand side of (5.2). The coefficients $\varepsilon(a)$ in (5.3) and $c(b)$ in (5.4) are determined from the self-consistency, namely, they are

$$(5.5) \quad c(b) = (-1)^{\frac{b-1}{2}}, \quad \varepsilon(a) = (-1)^{\frac{a-1}{2}} \frac{i}{\nu} \cot \frac{\pi}{2\nu} (\nu\alpha + a).$$

Let us see why this prescription is good. Consider the formula (5.2) together with several identities.

First,

$$\frac{D_{2n-1}(\alpha + 2m\frac{1-\nu}{\nu})}{D_{2(m+n)-1}(\alpha)} = \Gamma(\nu)^{\frac{2m}{\nu}} (1-\nu)^{-m} \prod_{j=1}^m \left(\frac{m+n-j}{\frac{\alpha}{2} - j + \frac{2(n+m)-1}{2\nu}} \right),$$

Further we have for $n > m$

$$\frac{D_{2n-1}(2-\alpha-2m\frac{1-\nu}{\nu})}{D_{2(n-m)-1}(2-\alpha)} = \Gamma(\nu)^{-\frac{2m}{\nu}}(1-\nu)^m \prod_{j=1}^m \left(\frac{j - \frac{\alpha}{2} + \frac{2(n-m)-1}{2\nu}}{n-j} \right),$$

and for $1 < n \leq m$

$$D_{2n-1}(2-\alpha-2m\frac{1-\nu}{\nu})D_{2(m-n)+1}(\alpha) = \Gamma(\nu)^{-\frac{2m}{\nu}}(1-\nu)^m(-1)^{m-n+1} \\ \times \frac{i}{\nu(n-1)!(m-n)!} \cot \frac{\pi}{2\nu} (\nu\alpha + 2(m-n) + 1) \prod_{j=1}^m \left(j - \frac{\alpha}{2} - \frac{1}{2\nu}(2(m-n)+1) \right).$$

For $A = (a_1, \dots, a_p), B = (b_1, \dots, b_p)$ we set

$$\mathcal{D}(A|B|\kappa, \alpha) = \det \left(\Theta \left(\frac{ia_j}{2\nu}, \frac{ib_k}{2\nu} | \kappa, \alpha \right) \right)_{j,k=1,\dots,p}.$$

Now, consider (5.2). Set

$$I^+ = (a_1, \dots, a_p), \quad I^- = (b_1, \dots, b_p), \quad I^-_{<} = (b_1, \dots, b_r), \quad I^-_{>} = (b_{r+1}, \dots, b_p),$$

where $a_1 < \dots < a_p$ and $b_1 < \dots < b_r < 2m < b_{r+1} < \dots < b_p$. Using the above formulae and (3.4), (3.6), and (5.5), one easily finds that (5.2) is equivalent to the compatibility condition

$$(5.6) \quad F \cdot \mathcal{D}(I^+|I^-|\kappa, \alpha + 2m\frac{1-\nu}{\nu}) \mathcal{D}(I_{\text{odd}}(m)|\nu(\alpha - 2I(m))|\kappa, \alpha) \\ = \mathcal{D}((I_{\text{odd}}(m) \setminus (2m - I^-_{<})) \sqcup (I^+ + 2m)|\nu(\alpha - 2I(m)) \sqcup (I^-_{>} - 2m)|\kappa, \alpha),$$

where

$$F = (-2)^{-rm}(-\nu)^{-r} \prod_{j=1}^p \prod_{k=1}^m \frac{a_j + 2m - (2k-1)}{a_j + 2m + \nu(\alpha - 2k)} \prod_{j=r+1}^p \prod_{k=1}^m \frac{b_j - 2m - \nu(\alpha - 2k)}{b_j - 2m + 2k - 1} \\ \times \prod_{j=1}^r \prod_{k=1}^m (2m - b_j + \nu(\alpha - 2k)) \cdot \prod_{j=1}^r \frac{1}{\left(\frac{b_j-1}{2}\right)! \left(m - \frac{b_j+1}{2}\right)!}.$$

It is already remarkable that F is independent of κ because it implies that the identity (5.6) is valid for $\kappa = \infty$, where the function $\Theta(l, m|\kappa, \alpha)$ reduces to $-\frac{i}{l+m}$. One can easily check the identity (5.6) for $\kappa = \infty$. This implies that the compatibility condition is equivalent to

$$(5.7) \quad \frac{\mathcal{D}(I^+|I^-|\kappa, \alpha + 2m\frac{1-\nu}{\nu})}{\mathcal{D}(I^+|I^-|\infty, \alpha + 2m\frac{1-\nu}{\nu})} \cdot \frac{\mathcal{D}(I_{\text{odd}}(m)|\nu(\alpha - 2I(m))|\kappa, \alpha)}{\mathcal{D}(I_{\text{odd}}(m)|\nu(\alpha - 2I(m))|\infty, \alpha)} \\ = \frac{\mathcal{D}((I_{\text{odd}}(m) \setminus (2m - I^-_{<})) \sqcup (I^+ + 2m)|\nu(\alpha - 2I(m)) \sqcup (I^-_{>} - 2m)|\kappa, \alpha)}{\mathcal{D}((I_{\text{odd}}(m) \setminus (2m - I^-_{<})) \sqcup (I^+ + 2m)|\nu(\alpha - 2I(m)) \sqcup (I^-_{>} - 2m)|\infty, \alpha)}.$$

We have checked (5.7) for $m = 1$ up to level 8 and κ^{-8} . We refer the reader to section 9 for the proof of similar identities in the sG case.

6. GLUING TWO CHIRALITIES

We have seen that, in the chiral CFT theory, the shifted primary field $\phi_{\alpha+2\frac{1-\nu}{\nu}}(0)$ is realised in \mathcal{H}_α as

$$(6.1) \quad \phi_{\alpha+2\frac{1-\nu}{\nu}}(0) \cong \beta_1^* \gamma_{\text{screen},1}^* \phi_\alpha(0).$$

The primary field in the left hand side is normalised by this formula. We have a similar formula for the second chirality,

$$(6.2) \quad \bar{\phi}_{\alpha+2\frac{1-\nu}{\nu}}(0) \cong \bar{\gamma}_1^* \bar{\beta}_{\text{screen},1}^* \bar{\phi}_\alpha(0).$$

Consider the primary field $\Phi_\alpha(0)$ in the full CFT normalised by the CFT theory. We have a relation

$$\Phi_\alpha(0) = S(\alpha) \phi_\alpha(0) \bar{\phi}_\alpha(0),$$

for some function $S(\alpha)$. We shall determine the exact ratio between $\Phi_{\alpha+2\frac{1-\nu}{\nu}}(0)$ and $\beta_1^* \gamma_{\text{screen},1}^* \bar{\gamma}_1^* \bar{\beta}_{\text{screen},1}^* \Phi_\alpha(0)$. This, together with the definitions (6.1), (6.2) and natural analyticity assumptions, allows one to determine $S(\alpha)$. However, we shall not write an explicit formula for $S(\alpha)$ because for the application to the OPE what we need are the ratios (see (6.9)).

The normalised three point function in full CFT can be extracted, for example, from the known results in Liouville theory. We quote from Appendix A, (A.1) and (A.2), which lead to the following formula:

$$(6.3) \quad \frac{\langle \Phi_{1-\kappa}(-\infty) \Phi_{\alpha+2\frac{1-\nu}{\nu}}(0) \Phi_{1+\kappa}(\infty) \rangle}{\langle \Phi_{1-\kappa}(-\infty) \Phi_\alpha(0) \Phi_{1+\kappa}(\infty) \rangle} = \mu^2 \Gamma(\nu)^2 \cdot Y(x) W(\alpha, \kappa) \bar{W}(\alpha, \kappa),$$

where

$$(6.4) \quad x = \frac{\alpha}{2} + \frac{1-\nu}{2\nu},$$

and we have set

$$\begin{aligned} Y(x) &= -2\nu x \cdot \frac{\Gamma^2(\nu x + 1/2 - \nu/2) \Gamma(\nu - 2\nu x)}{\Gamma^2(1/2 + \nu/2 - \nu x) \Gamma(2\nu x + 1 - \nu)} \cdot \frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)}, \\ W(\alpha, \kappa) &= \frac{\Gamma(\alpha\nu/2 - \nu + 1 + \kappa\nu)}{\Gamma(-\alpha\nu/2 + \nu + \kappa\nu)}, \\ \bar{W}(\alpha, \kappa) &= W(\alpha, -\kappa). \end{aligned}$$

Now let us compare this result with the corresponding one in the chiral theory. Using (3.6), we immediately obtain:

$$\begin{aligned} & \frac{\langle 1-\kappa | \beta_1^* \gamma_{\text{screen},1}^* \phi_\alpha(0) | 1+\kappa \rangle}{\langle 1-\kappa | \phi_\alpha(0) | 1+\kappa \rangle} = e^{\frac{\pi i}{2}(2\nu - \alpha\nu - 1)} X(x) \\ & \times (\nu\kappa)^{\alpha\nu - 2\nu + 1} \left(1 - \frac{\alpha}{2} - \frac{1}{2\nu} \right) \Theta\left(\frac{i}{2\nu}, -i\left(1 - \frac{\alpha}{2}\right) \middle| \kappa, \alpha \right), \end{aligned}$$

where

$$X(x) = -i\Gamma(\nu)^{-2x+1} \cdot \frac{\Gamma(x+1/2)}{\Gamma(x)} \cdot 2^{-2\nu x + \nu} \frac{\Gamma(-\nu x + \nu/2)}{\Gamma(-\nu x + \nu/2 + 1/2)}.$$

The first thing to consider is the essential, κ -dependent, function $W(\alpha, \kappa)$. The asymptotics of $W(\alpha, \kappa)$ for $\kappa \rightarrow \infty$ is given by

$$W(\alpha, \kappa) \simeq (\kappa\nu)^{\alpha\nu-2\nu+1} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} \frac{1}{(\kappa\nu)^{2k}} B_{2k+1}(\alpha\nu/2 - \nu + 1)\right).$$

where $B_n(z)$ denotes the Bernoulli polynomial. On the other hand, using the asymptotic expansion of $\Theta(l, m|\kappa, \alpha)$ available up to κ^{-8} , we have checked that to this order

$$(6.5) \quad \left(1 - \frac{\alpha}{2} - \frac{1}{2\nu}\right) \Theta\left(\frac{i}{2\nu}, -i\left(1 - \frac{\alpha}{2}\right) \middle| \kappa, \alpha\right) \\ \simeq \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} \frac{1}{(\kappa\nu)^{2k}} B_{2k+1}(\alpha\nu/2 - \nu + 1)\right).$$

So, we see that the κ -dependence agrees between the chiral three point functions computed in our way with the one computed in CFT:

$$(6.6) \quad \frac{\langle 1 - \kappa | \beta_1^* \gamma_{\text{screen},1}^* \phi_\alpha(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \phi_\alpha(0) | 1 + \kappa \rangle} = e^{\frac{\pi i}{2}(2\nu - \alpha\nu - 1)} X(x) W(\alpha, \kappa).$$

Similarly we have for the second chirality that

$$(6.7) \quad \frac{\langle 1 - \kappa | \bar{\beta}_{\text{screen},1}^* \bar{\gamma}_1^* \bar{\phi}_\alpha(0) | 1 + \kappa \rangle}{\langle 1 - \kappa | \bar{\phi}_\alpha(0) | 1 + \kappa \rangle} = e^{-\frac{\pi i}{2}(2\nu - \alpha\nu - 1)} X(x) \bar{W}(\alpha, \kappa).$$

Combining (6.3) with (6.6), (6.7) we obtain:

$$\Phi_{\alpha+2\frac{1-\nu}{\nu}}(0) \cong C_1(\alpha) \beta_1^* \bar{\gamma}_1^* \Phi_\alpha^{(1)}(0),$$

where

$$\Phi_\alpha^{(1)}(0) = i\mu^2 \cot \frac{\pi\nu}{2}(2 - \alpha) \bar{\beta}_{\text{screen},1}^* \gamma_{\text{screen},1}^* \Phi_\alpha(0)$$

is one time screened primary field, and

$$C_1(\alpha) = \Gamma(\nu)^2 Y(x) X(x)^{-2} i \cot \pi(\nu x - \frac{\nu}{2}),$$

there is a change of sign coming from different order of fermions in the formulae (6.2) and (6.7).

After simplification, we arrive at the result

$$(6.8) \quad C_1(\alpha) = -\nu \Gamma(\nu)^{4x} \frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)} \cdot \frac{\Gamma(x)}{\Gamma(x+1/2)} \cdot \frac{\Gamma(-x+1/2)}{\Gamma(-x)} i \cot \pi x,$$

where x is defined in (6.4). Being just a normalisation of primary fields, this formula may not look very significant. Nevertheless, we shall see that the formula (6.8) implies the Lukyanov-Zamolodchikov formula for the one-point functions of primary fields [9] (see subsection 10.1). We remark that

$$(6.9) \quad \frac{S(\alpha + 2\frac{1-\nu}{\nu})}{S(\alpha)} = i\mu^2 \cot \frac{\pi\nu}{2}(2 - \alpha) C_1(\alpha).$$

It is easy to generalise the calculation given above. For any $m > 0$ we have

$$(6.10) \quad \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \cong C_m(\alpha) \beta_{I_{\text{odd}}(m)}^* \bar{\gamma}_{I_{\text{odd}}(m)}^* \Phi_{\alpha}^{(m)}(0),$$

where

$$(6.11) \quad \Phi_{\alpha}^{(m)}(0) = i^m \mu^{2m} \prod_{j=0}^{m-1} \cot \frac{\pi\nu}{2} (2j - \alpha) \bar{\beta}_{\text{screen}, I(m)}^* \gamma_{\text{screen}, I(m)}^* \Phi_{\alpha}(0),$$

$$C_m(\alpha) = \prod_{j=0}^{m-1} C_1(\alpha + 2j\frac{1-\nu}{\nu}).$$

The multiplier is included in order that the one-point function of $\Phi_{\alpha}^{(m)}(0)$ be a simple power in μ in the sG case (see (8.3)). The formula (6.10) generalises (4.2) to the case of two chiralities. The entire space $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}} \otimes \bar{\mathcal{V}}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$ is embedded into the space $\mathcal{H}_{\alpha} \otimes \bar{\mathcal{H}}_{\alpha}$ the basis being

$$\beta_{I^+}^* \bar{\beta}_{I^+}^* \bar{\gamma}_{I^-}^* \gamma_{I^-}^* \Phi_{\alpha}^{(m)}(0).$$

The identification of descendants (see (5.2)) in the case of two chiralities reads as

$$(6.12) \quad \beta_{I^+}^* \bar{\beta}_{I^+}^* \bar{\gamma}_{I^-}^* \gamma_{I^-}^* \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \\ \cong C_m(\alpha) \beta_{I^++2m}^* \bar{\beta}_{I^+-2m}^* \bar{\gamma}_{I^--2m}^* \gamma_{I^--2m}^* \beta_{I_{\text{odd}}(m)}^* \bar{\gamma}_{I_{\text{odd}}(m)}^* \Phi_{\alpha}^{(m)}(0).$$

7. CREATION OPERATORS IN SG CASE AND FUNCTION $\omega_R^{\text{sG}}(\zeta, \xi | \alpha)$

Before embarking upon the scaling limit to the sG model, let us give a brief review about the creation operators in the inhomogeneous case. For the inhomogeneous model the annihilation operators split into two parts:

$$\mathbf{b}(\zeta) = \mathbf{b}^+(\zeta) + \mathbf{b}^-(\zeta), \quad \mathbf{c}(\zeta) = \mathbf{c}^+(\zeta) + \mathbf{c}^-(\zeta),$$

$$\mathbf{b}^{\pm}(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 \zeta_0^{\mp 2} - 1)^{-p} \mathbf{b}_p^{\pm}, \quad \mathbf{c}^{\pm}(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 \zeta_0^{\mp 2} - 1)^{-p} \mathbf{c}_p^{\pm}.$$

Consider the creation operator $\mathbf{b}_0^*(\zeta)$. The local operators are created by its power series at $\zeta^2 = \zeta_0^{\pm 2}$:

$$\mathbf{b}_0^*(\zeta) \underset{\zeta^2 \rightarrow \zeta_0^{\pm 2}}{\cong} \sum_{p=1}^{\infty} (\zeta^2 \zeta_0^{\mp 2} - 1)^{p-1} \mathbf{b}_{0,p}^{\pm*},$$

we denote the corresponding sums by

$$\mathbf{b}_0^{\pm*}(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 \zeta_0^{\mp 2} - 1)^{p-1} \mathbf{b}_{0,p}^{\pm*}.$$

Then we introduce the notation

$$\Omega_0^{\epsilon\epsilon'} = \frac{4}{(2\pi i)^2} \int_{\Gamma_{\epsilon}} \int_{\Gamma_{\epsilon'}} \omega(\zeta/\xi, \alpha) \mathbf{b}^{\epsilon}(\zeta) \mathbf{c}^{\epsilon'}(\xi) \frac{d\zeta^2}{\zeta^2} \frac{d\xi^2}{\xi^2},$$

and define

$$(7.1) \quad \begin{aligned} \mathbf{b}^{+*}(\zeta) &= e^{-\Omega_0^{++}} \mathbf{b}_0^{+*}(\zeta) e^{\Omega_0^{++}}, & \mathbf{c}^{+*}(\zeta) &= e^{-\Omega_0^{++}} \mathbf{c}_0^{+*}(\zeta) e^{\Omega_0^{++}}, \\ \mathbf{b}^{-*}(\zeta) &= e^{-\Omega_0^{--}} \mathbf{b}_0^{-*}(\zeta) e^{\Omega_0^{--}}, & \mathbf{c}^{-*}(\zeta) &= e^{-\Omega_0^{--}} \mathbf{c}_0^{-*}(\zeta) e^{\Omega_0^{--}}. \end{aligned}$$

The functional $Z_{\mathbf{n}}^{-s}$ computed on the descendants created by these operators takes the determinant form [5, 6] with the pairing being:

$$(7.2) \quad \begin{aligned} Z_{\mathbf{n}}^{-s} \{ \mathbf{b}^{+*}(\zeta) \mathbf{c}^{+*}(\xi) (q^{2\alpha S(0)}) \} &= \omega_{\mathbf{n}}(\zeta, \xi | \alpha, -s), \\ Z_{\mathbf{n}}^{-s} \{ \mathbf{b}^{+*}(\zeta) \mathbf{c}^{-*}(\xi) (q^{2\alpha S(0)}) \} &= \omega_{\mathbf{n}}(\zeta, \xi | \alpha, -s) + \omega_0(\zeta/\xi, \alpha), \\ Z_{\mathbf{n}}^{-s} \{ \mathbf{b}^{-*}(\zeta) \mathbf{c}^{+*}(\xi) (q^{2\alpha S(0)}) \} &= \omega_{\mathbf{n}}(\zeta, \xi | \alpha, -s) + \omega_0(\zeta/\xi, \alpha), \\ Z_{\mathbf{n}}^{-s} \{ \mathbf{b}^{-*}(\zeta) \mathbf{c}^{-*}(\xi) (q^{2\alpha S(0)}) \} &= \omega_{\mathbf{n}}(\zeta, \xi | \alpha, -s). \end{aligned}$$

We do not give the definition of the function $\omega_{\mathbf{n}}(\zeta, \xi | \alpha, -s)$ which is easy to find from [1] because we shall be interested only in its scaling limit.

We conjecture that the following scaling limit exists for the operators $\mathbf{b}^{\pm*}(\zeta)$, $\mathbf{c}^{\pm*}(\zeta)$:

$$\begin{aligned} \frac{1}{2} \mathbf{b}^{+*}(\zeta) &\xrightarrow{\text{scaling}} \beta^{+*}(\zeta) \simeq \beta^*(\mu\zeta) + \bar{\beta}_{\text{screen}}^*(\zeta/\mu), & \zeta \rightarrow \infty, \\ \frac{1}{2} \mathbf{c}^{+*}(\zeta) &\xrightarrow{\text{scaling}} \gamma^{+*}(\zeta) \simeq \gamma^*(\mu\zeta) + \bar{\gamma}_{\text{screen}}^*(\zeta/\mu), & \zeta \rightarrow \infty, \\ \frac{1}{2} \mathbf{b}^{-*}(\zeta) &\xrightarrow{\text{scaling}} \beta^{-*}(\zeta) \simeq \bar{\beta}^*(\zeta/\mu) + \beta_{\text{screen}}^*(\mu\zeta), & \zeta \rightarrow 0, \\ \frac{1}{2} \mathbf{c}^{-*}(\zeta) &\xrightarrow{\text{scaling}} \gamma^{-*}(\zeta) \simeq \bar{\gamma}^*(\zeta/\mu) + \gamma_{\text{screen}}^*(\mu\zeta), & \zeta \rightarrow 0. \end{aligned}$$

These operators have the asymptotics (3.1), (3.2), (3.13), (3.14). An explanation to this conjecture is given in [3]. Notice that the appearance of μ in the right hand sides is not a part of the conjecture, but is a corollary of the computations done in the conformal case [1]: the key identity is

$$\lim \zeta_0 (Ca)^\nu = \mu^{-1}.$$

In this section we define our main function $\omega_R^{\text{sG}}(\zeta, \xi | \alpha)$ as the scaling limit of $\omega_{\mathbf{n}}(\zeta, \xi | \alpha, s)$ for $\alpha = 2s \frac{1-\nu}{\nu}$, and as the analytic continuation with respect to α in general. We shall be rather sketchy because the construction repeats very much what has been done in the conformal case [1].

We start with the DDV equation:

$$(7.3) \quad \frac{1}{i} \log \mathbf{a}(\zeta) = \pi M R(\zeta^{1/\nu} - \zeta^{-1/\nu}) - 2 \text{Im} \int_0^\infty R(\zeta/\xi) \log(1 + \mathbf{a}(\xi e^{+i0})) \frac{d\xi^2}{\xi^2},$$

where as usual it is convenient to define $R(\zeta)$ through a more general object:

$$\begin{aligned} R(\zeta, \alpha) &= \int_{-\infty}^{\infty} \zeta^{2ik} \widehat{R}(k, \alpha) \frac{dk}{2\pi}, & \widehat{R}(k, \alpha) &= \frac{\sinh \pi((2\nu - 1)k - i\alpha/2)}{2 \sinh \pi((1 - \nu)k + i\alpha/2) \cosh(\pi\nu k)}, \\ R(\zeta) &= R(\zeta, 0). \end{aligned}$$

Notice that

$$(7.4) \quad \widehat{R}(k, -\alpha) = \widehat{R}(-k, \alpha), \quad \widehat{R}(k, \alpha + 2) = \widehat{R}(k, \alpha).$$

We hope using the same letter for the resolvent as in the conformal case [1] is not very confusing. Similarly to [1], we write the equation for the resolvent:

$$(7.5) \quad R_{\text{dress}} + R * R_{\text{dress}} = R,$$

and define ω_R^{sG} by

$$(7.6) \quad \frac{1}{2\pi i} \omega_R^{\text{sG}} = -F^+ * F^- + F^+ * R_{\text{dress}} * F^-,$$

where

$$f * g = \int_0^\infty f(\zeta)g(\zeta)dm(\zeta), \quad dm(\zeta) = 2\text{Re} \left(\frac{1}{1 + \mathbf{a}(\zeta e^{-i0})} \right) \frac{d\zeta^2}{\zeta^2},$$

and

$$F^\pm(\zeta, \xi) = F^\pm(\zeta/\xi), \quad F^\pm(\zeta) = \mp \int \frac{dk}{2\pi} \zeta^{2ik} \frac{e^{\mp \pi \nu k}}{2 \cosh(\pi \nu k)}.$$

It is very convenient to consider not the function $\omega_R^{\text{sG}}(\zeta, \xi|\alpha)$, but rather its Mellin transform. Namely, we introduce $\Theta_R^{\text{sG}}(l, m|\alpha)$ by

$$R_{\text{dress}}(\zeta, \xi) - R(\zeta/\xi, \alpha) = \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{dl}{2\pi} \frac{dm}{2\pi} \widehat{R}(l, \alpha) \Theta_R^{\text{sG}}(l, m|\alpha) \widehat{R}(m, -\alpha) \zeta^{2il} \xi^{2im}.$$

Rewriting (7.5) we get the equation for Θ_R^{sG} :

$$(7.7) \quad \Theta_R^{\text{sG}}(l, m|\alpha) + G(l + m) + \int_{-\infty}^\infty G(l - k) \widehat{R}(k, \alpha) \Theta_R^{\text{sG}}(k, m|\alpha) \frac{dk}{2\pi} = 0,$$

where $G(k)$ is the moment of our measure:

$$G(k) = \int_0^\infty \zeta^{-2ik} dm(\zeta).$$

There are no obstacles for the convergence of the integral in the entire complex plane of k , so, $G(k)$ is an entire function.

Now it is rather easy to see that

$$(7.8) \quad \omega_R^{\text{sG}}(\zeta, \xi|\alpha) = -\frac{\pi i}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{dl}{2\pi} \frac{dm}{2\pi} \zeta^{2il} \xi^{2im} \frac{e^{-\pi \nu l}}{\cosh(\pi \nu l)} \Theta_R^{\text{sG}}(l, m|\alpha) \frac{e^{-\pi \nu m}}{\cosh(\pi \nu m)}.$$

What we are really interested in are the coefficients in the asymptotic expansion

$$(7.9) \quad \omega_R^{\text{sG}}(\zeta, \xi|\alpha) \simeq \sum_{j,k=1}^\infty \zeta^{-\epsilon_1 \frac{2j-1}{\nu}} \xi^{-\epsilon_2 \frac{2k-1}{\nu}} \omega_{R_{\epsilon_1(2j-1), \epsilon_2(2k-1)}}^{\text{sG}}(\alpha) \quad (\zeta^{\epsilon_1}, \xi^{\epsilon_2} \rightarrow \infty),$$

where $\epsilon_1, \epsilon_2 = \pm 1$. Obviously,

$$\omega_{R_{2j-1, 2k-1}}^{\text{sG}}(\alpha) = \text{sgn}(2j-1) \text{sgn}(2k-1) \frac{i}{2\pi \nu^2} \Theta_R^{\text{sG}}\left(\frac{2j-1}{2\nu} i, \frac{2k-1}{2\nu} i|\alpha\right),$$

for all odd integer $2j - 1$, $2k - 1$.

The function $\mathbf{a}(\zeta)$ possesses the symmetry

$$(7.10) \quad \mathbf{a}(\zeta) = (\mathbf{a}(\zeta^{-1}))^{-1}.$$

This property implies that $G(k)$ is an even function. Together with

$$\hat{R}(k, 2 - \alpha) = \hat{R}(-k, \alpha),$$

which follows from (7.4), we then obtain the symmetry

$$(7.11) \quad \Theta_R^{\text{sG}}(l, m|2 - \alpha) = \Theta_R^{\text{sG}}(-l, -m|\alpha).$$

The position of the singularities is important in the derivation. This symmetry property provides, for example, the invariance of our main formula (8.2) derived in the next section under the interchange of the two chiralities.

Notice that the symmetry (7.10) is a property specific to the maximal eigenvalue of the Matsubara transfer-matrix. Certainly, this is the only case interesting for us, but most of our computations hold true if we consider $\mathbf{a}(\zeta)$ corresponding to other eigenvalues as well. In the latter case the symmetry (7.10) is broken. Since considering other eigenvalues may be of physical interest, in the following calculations we shall not use the fact that $G(k)$ is even.

8. MAIN FORMULA

Following [3] we conclude that the one-point functions at finite R are described by usual fermionic formulae. For the same chiralities we have to take $\omega_R^{\text{sG}}(\zeta, \xi|\alpha)$ while for different chiralities we have to take $\omega_R^{\text{sG}}(\zeta, \xi|\alpha) + \omega_0(\zeta/\xi, \alpha)$:

$$\begin{aligned} \frac{\langle \beta^{+*}(\zeta) \gamma^{+*}(\xi) \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} &= \omega_R^{\text{sG}}(\zeta, \xi|\alpha), \\ \frac{\langle \beta^{+*}(\zeta) \gamma^{-*}(\xi) \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} &= \omega_R^{\text{sG}}(\zeta, \xi|\alpha) + \omega_0(\zeta/\xi, \alpha), \\ \frac{\langle \beta^{-*}(\zeta) \gamma^{+*}(\xi) \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} &= \omega_R^{\text{sG}}(\zeta, \xi|\alpha) + \omega_0(\zeta/\xi, \alpha), \\ \frac{\langle \beta^{-*}(\zeta) \gamma^{-*}(\xi) \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} &= \omega_R^{\text{sG}}(\zeta, \xi|\alpha). \end{aligned}$$

We have computed the asymptotics of $\omega_R^{\text{sG}}(\zeta, \xi|\alpha)$ in the previous section. Now we compute the asymptotics of $\omega_0(\zeta, \alpha)$:

(8.1)

$$\begin{aligned} \omega_0(\zeta, \alpha) &= -i \int_{-\infty}^{\infty} \zeta^{2ik} \frac{\sinh \frac{\pi}{2}(2(1-\nu)k + i\alpha)}{2 \sinh \frac{\pi}{2}(2k + i\alpha) \cosh \pi \nu k} dk \\ &\simeq \frac{i\epsilon}{\nu} \sum_{j=1}^{\infty} \zeta^{-\frac{2j-1}{\nu}\epsilon} \cot \frac{\pi}{2\nu}(\nu\alpha + (2j-1)\epsilon) + i\epsilon \sum_{j=1}^{\infty} \zeta^{\alpha-1-(2j-1)\epsilon} \tan \frac{\pi\nu}{2}(\alpha-1-(2j-1)\epsilon), \end{aligned}$$

for $\zeta^\epsilon \rightarrow \infty$.

From the asymptotics one can read the expectation values of the fermion operators. There is a difference between the CFT case and the sG case. Two cases have different selection rules. In the CFT case, the expectation values are zero between chiral and anti-chiral components, while there are non-zero values between screening and non-screening operators. This is opposite in the sG case. Let us list some of the non-zero expectation values in the sG case. It is convenient to introduce the convention: for $j \geq 1$

$$\tilde{\beta}_{2j-1}^* = \beta_{2j-1}^*, \quad \tilde{\beta}_{1-2j}^* = \bar{\beta}_{2j-1}^*, \quad \tilde{\gamma}_{2j-1}^* = \gamma_{2j-1}^*, \quad \tilde{\gamma}_{1-2j}^* = \bar{\gamma}_{2j-1}^*.$$

Then, for $a, b \in 2\mathbb{Z} + 1$, we have

$$\begin{aligned} & \frac{\langle \tilde{\beta}_a^* \tilde{\gamma}_b^* \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} \\ &= \text{sgn}(a) \text{sgn}(b) \frac{i\mu^{\frac{a+b}{\nu}}}{2\pi\nu^2} \left(\Theta_R^{\text{sG}} \left(\frac{ia}{2\nu}, \frac{ib}{2\nu} | \alpha \right) - \text{sgn}(a) \delta_{a,-b} 2\pi\nu \cot \frac{\pi\nu}{2} (\nu\alpha + a) \right), \end{aligned}$$

and for $j, k \geq 1$

$$\frac{\langle \bar{\beta}_{\text{screen},j}^* \gamma_{\text{screen},k}^* \Phi_\alpha(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} = i\mu^{\alpha-2j} \delta_{j,k} \tan \frac{\pi\nu}{2} (\alpha - 2j).$$

Similarly to the conformal case let us introduce for $\#(A) = \#(B) = n$:

$$\begin{aligned} \mathcal{D}_R^{\text{sG}}(A|B|\alpha) &= \prod_{j=1}^n \text{sgn}(a_j) \text{sgn}(b_j) \left(\frac{i}{2\pi\nu^2} \right)^n \det(D_{a_n, b_k}(\alpha))|_{j,k=1, \dots, n}, \\ D_{a,b}(\alpha) &= \Theta_R^{\text{sG}} \left(\frac{ia}{2\nu}, \frac{ib}{2\nu} | \alpha \right) - \delta_{a,-b} \text{sgn}(a) 2\pi\nu \cot \frac{\pi\nu}{2} (\nu\alpha + a), \end{aligned}$$

We have seen that the primary fields $\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)$ and all their descendants are obtained from

$$\beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha^{(m)}(0).$$

So is sufficient to write down the formula for one-point functions of these operators in the sG model. This is immediate. The only important thing to notice is that the contribution from screening operators completely disappears as a result of our normalisation of $\Phi^{(m)}(0)$. Thus we have the main formula:

$$\begin{aligned} (8.2) \quad & \frac{\langle \beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_\alpha^{(m)}(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} \\ &= \mu^{2m\alpha-2m^2+\frac{1}{\nu}(|I^+|+|I^-|+|\bar{I}^+|+|\bar{I}^-|)} \mathcal{D}_R^{\text{sG}}(I^+ \cup (-\bar{I}^+) | I^- \cup (-\bar{I}^-) | \alpha), \end{aligned}$$

with the requirements $\#(I^+) = \#(I^-) + m$, $\#(\bar{I}^+) + m = \#(\bar{I}^-)$. In particular, we have

$$(8.3) \quad \frac{\langle \Phi_\alpha^{(m)}(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} = \mu^{2m\alpha-2m^2}.$$

9. PROOF OF COMPATIBILITY

The main formula (8.2) leads to the evaluation of the expectation values for the primary fields and the descendants. Namely, from (6.10) and (6.12), we obtain

$$\begin{aligned}
 (9.1) \quad & \frac{\langle \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \rangle_R^{\text{sG}}}{\langle \Phi_{\alpha}(0) \rangle_R^{\text{sG}}} = C_m(\alpha) \frac{\langle \beta_{I_{\text{odd}}(m)}^* \bar{\gamma}_{I_{\text{odd}}(m)}^* \Phi_{\alpha}^{(m)}(0) \rangle_R^{\text{sG}}}{\langle \Phi_{\alpha}(0) \rangle_R^{\text{sG}}}, \\
 & \frac{\langle \beta_{I^+}^* \bar{\beta}_{\bar{I}^+}^* \bar{\gamma}_{\bar{I}^-}^* \gamma_{I^-}^* \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \rangle_R^{\text{sG}}}{\langle \Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \rangle_R^{\text{sG}}} \\
 & = \frac{\langle \beta_{I^++2m}^* \bar{\beta}_{\bar{I}^+-2m}^* \bar{\gamma}_{\bar{I}^--2m}^* \gamma_{I^--2m}^* \beta_{I_{\text{odd}}(m)}^* \bar{\gamma}_{I_{\text{odd}}(m)}^* \Phi_{\alpha}^{(m)}(0) \rangle_R^{\text{sG}}}{\langle \beta_{I_{\text{odd}}(m)}^* \bar{\gamma}_{I_{\text{odd}}(m)}^* \Phi_{\alpha}^{(m)}(0) \rangle_R^{\text{sG}}},
 \end{aligned}$$

where $\#(I^+) = \#(I^-)$ and $\#(\bar{I}^+) = \#(\bar{I}^-)$. In order for these equalities to hold, certain consistency conditions need to be satisfied. Recalling the definition (5.3), (5.4), (5.5), we find that the conditions read respectively as follows.

$$(9.2) \quad \det(D_{a,b}(0, \alpha))_{a \in I_{\text{odd}}(m), b \in -I_{\text{odd}}(m)} = \prod_{j=0}^{m-1} D_{1,-1}(j, \alpha),$$

$$\begin{aligned}
 (9.3) \quad & \det(D_{a,b}(0, \alpha))_{a \in J^+, b \in J^-} = (-1)^{\#} \prod_{a \in (2m-I^-)_{>} \sqcup (2m-\bar{I}^+)_{>}} 2\pi\nu \cot \frac{\pi}{2\nu}(\nu\alpha + a) \\
 & \times \det(D_{a,b}(m, \alpha))_{\substack{a \in I^+ \cup (-\bar{I}^+) \\ b \in I^- \cup (-\bar{I}^-)}} \prod_{j=0}^{m-1} D_{1,-1}(j, \alpha),
 \end{aligned}$$

where

$$\begin{aligned}
 D_{a,b}(j, \alpha) &= D_{a,b}(\alpha + 2j\frac{1-\nu}{\nu}), \\
 J^+ &= (I^+ + 2m) \cup (-\bar{I}^+ + 2m)_{<} \cup (I_{\text{odd}}(m) \setminus (2m - I^-)_{>}), \\
 J^- &= (I^- - 2m)_{>} \cup -(\bar{I}^- + 2m) \cup (-I_{\text{odd}}(m) \setminus (\bar{I}^+ - 2m)_{<}),
 \end{aligned}$$

and the power of -1 can be easily computed from the fermionic commutation relations. These identities are understood as analytic continuation from the region

$$\alpha + 2m\frac{1-\nu}{\nu} < 2.$$

We show below that they are consequences of the single identity

$$(9.4) \quad \Theta_R^{\text{sG}}(l, j | \alpha + 2\frac{1-\nu}{\nu}) - \Theta_R^{\text{sG}}(l + \frac{i}{\nu}, j - \frac{i}{\nu} | \alpha) = - \frac{\Theta_R^{\text{sG}}(l + \frac{i}{\nu}, -\frac{i}{2\nu} | \alpha) \Theta_R^{\text{sG}}(\frac{i}{2\nu}, j - \frac{i}{\nu} | \alpha)}{\Theta_R^{\text{sG}}(\frac{i}{2\nu}, -\frac{i}{2\nu} | \alpha) - 2\pi\nu \cot \frac{\pi}{2}(\alpha + \frac{1}{\nu})},$$

where we start from the case when both α and $\alpha + 2\frac{1-\nu}{\nu}$ are inside the interval $(0, 2)$ and then continue analytically. First let us verify (9.4). We start from the defining

equation (7.7) with shifted α ,

$$\begin{aligned} & \Theta_R^{sG}(l, j | \alpha + 2\frac{1-\nu}{\nu}) + G(l + j) \\ & + \int_{-\infty}^{\infty} G(l - k) \hat{R}(k, \alpha + 2\frac{1-\nu}{\nu}) \Theta_R^{sG}(k, j | \alpha + 2\frac{1-\nu}{\nu}) \frac{dk}{2\pi} = 0. \end{aligned}$$

Noting the relation

$$\hat{R}(k, \alpha + 2\frac{1-\nu}{\nu}) = \hat{R}(k + \frac{i}{\nu}, \alpha),$$

we shift the contour to $\text{Im } k = -1/\nu$. Under our assumption, the only pole encountered on the way is $k = -i/2\nu$. Denote by $X(l, j)$ the left hand side of (9.4). Combining the above calculation with (7.7) we find

$$\begin{aligned} & X(l, j) + \int_{-\infty}^{\infty} G(l - k + \frac{i}{\nu}) \hat{R}(k, \alpha) X(k - \frac{i}{\nu}, j) \frac{dk}{2\pi} \\ & + \frac{1}{2\pi\nu} \tan \frac{\pi}{2}(\alpha + \frac{1}{\nu}) G(l + \frac{i}{2\nu}) \Theta_R^{sG}(-\frac{i}{2\nu}, j | \alpha + 2\frac{1-\nu}{\nu}) = 0, \end{aligned}$$

which can be solved as

$$X(l, j) = \frac{1}{2\pi\nu} \tan \frac{\pi}{2}(\alpha + \frac{1}{\nu}) \Theta_R^{sG}(l + \frac{i}{\nu}, -\frac{i}{2\nu} | \alpha) \Theta_R^{sG}(-\frac{i}{2\nu}, j | \alpha + 2\frac{1-\nu}{\nu}).$$

Setting $l = -i/2\nu$ in the last formula and eliminating $\Theta_R^{sG}(-\frac{i}{2\nu}, j | \alpha + 2\frac{1-\nu}{\nu})$, we arrive at (9.4).

Now we return to (9.2), (9.3). Specialising (9.4) to $l = ia/2\nu$ and $j = ib/2\nu$, we obtain

(9.5)

$$D_{a,b}(1, \alpha) D_{1,-1}(0, \alpha) = \det \begin{pmatrix} D_{1,-1}(0, \alpha) & D_{1,b-2}(0, \alpha) \\ D_{a+2,-1}(0, \alpha) & D_{a+2,b-2}(0, \alpha) \end{pmatrix}, \quad a \neq -1, \quad b \neq 1,$$

(9.6)

$$D_{-1,b}(1, \alpha) D_{1,-1}(0, \alpha) = -2\pi\nu \cot \frac{\pi}{2\nu}(\nu\alpha + 1) D_{1,b-2}(0, \alpha), \quad b \neq 1,$$

(9.7)

$$D_{a,1}(1, \alpha) D_{1,-1}(0, \alpha) = -2\pi\nu \cot \frac{\pi}{2\nu}(\nu\alpha + 1) D_{a+2,-1}(0, \alpha), \quad a \neq -1,$$

(9.8)

$$D_{-1,1}(1, \alpha) D_{1,-1}(0, \alpha) = -\left(2\pi\nu \cot \frac{\pi}{2\nu}(\nu\alpha + 1)\right)^2.$$

The equation (9.5) for $a = 1, b = -1$ is nothing but (9.2) for $m = 2$. By induction, (9.2) for general m reduces to (9.5) and the elementary identity of determinants

$$(9.9) \quad A_{1,1}^{m-2} \det(A_{i,j})_{1 \leq i,j \leq m} = \det \left(\begin{vmatrix} A_{1,1} & A_{1,j} \\ A_{i,1} & A_{i,j} \end{vmatrix} \right)_{2 \leq i,j \leq m}.$$

Consider now (9.3) for $m = 1$. If $1 \notin I^- \cup \bar{I}^+$, then (9.3) is a consequence of (9.5) and (9.9). Else contractions occur and one has to use together with (9.5) the identities (9.6)-(9.8). This is rather straightforward. The case of general m then follows by induction.

Now we would like to discuss an analogue of the main formula (8.2) for the case of negative m . Consider first of all the formula (9.8). Supposing that $\alpha, \alpha - 2\frac{1-\nu}{\nu} \in (0, 2)$ we can rewrite it as

$$D_{-1,1}(0, \alpha) D_{1,-1}(-1, \alpha) = - \left(2\pi\nu \cot \frac{\pi}{2} \left(\alpha - \frac{1}{\nu} \right) \right)^2.$$

This identity implies

$$\frac{\langle \Phi_{\alpha-2\frac{1-\nu}{\nu}}(0) \rangle_R^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\text{sG}}} = -C_{-1}(\alpha) \mu^{-2\alpha+2\frac{1-\nu}{\nu}} \frac{i}{2\pi\nu^2} D_{-1,1}(0, \alpha).$$

where we define $C_{-m}(\alpha)$ by the equation

$$(9.10) \quad C_{-m}(\alpha) C_m(\alpha - 2m\frac{1-\nu}{\nu}) = \nu^{2m} \prod_{j=1}^m \tan^2 \frac{\pi}{2} \left(\alpha - \frac{j}{\nu} \right).$$

Generally, it is not hard to derive from (9.4) another identity:

$$(9.11) \quad \begin{aligned} & \Theta_R^{\text{sG}}(l, j | \alpha - 2\frac{1-\nu}{\nu}) - \Theta_R^{\text{sG}}(l - \frac{j}{\nu}, j + \frac{j}{\nu} | \alpha) \\ &= - \frac{\Theta_R^{\text{sG}}(l - \frac{j}{\nu}, \frac{j}{2\nu} | \alpha) \Theta_R^{\text{sG}}(-\frac{j}{2\nu}, j + \frac{j}{\nu} | \alpha)}{2\pi\nu \cot \frac{\pi}{2} \left(\alpha - \frac{1}{\nu} \right) + \Theta_R^{\text{sG}}(-\frac{j}{2\nu}, \frac{j}{2\nu} | \alpha)}, \end{aligned}$$

which is understood as analytical continuation from the region $\alpha, \alpha - 2\frac{1-\nu}{\nu} \in (0, 2)$. From (9.11) we obtain analogues of (9.5)-(9.8), and further of (9.2), (9.3):

$$\begin{aligned} \det (D_{a,b}(0, \alpha))_{a \in -I_{\text{odd}}(m), b \in I_{\text{odd}}(m)} &= \prod_{j=0}^{m-1} D_{-1,1}(-j, \alpha), \\ \det (D_{a,b}(0, \alpha))_{a \in J^+, b \in J^-} &= (-1)^{\#} \prod_{a \in (2m-I^+)_> \sqcup (2m-\bar{I}^-)_>} 2\pi\nu \cot \frac{\pi}{2\nu} (\nu\alpha - a) \\ &\quad \times \det (D_{a,b}(m, \alpha))_{\substack{a \in I^+ \cup (-\bar{I}^+) \\ b \in I^- \cup (-\bar{I}^-)}} \prod_{j=0}^{m-1} D_{-1,1}(-j, \alpha), \end{aligned}$$

where

$$\begin{aligned} J^+ &= (I^+ - 2m)_> \cup (-\bar{I}^+ - 2m) \cup (I_{\text{odd}}(m) \setminus (2m - \bar{I}^-)_>), \\ J^- &= (I^- + 2m) \cup (-\bar{I}^- + 2m)_< \cup (-I_{\text{odd}}(m) \setminus (I^+ - 2m)_<), \end{aligned}$$

where $m > 0$, $\#(I^-) = \#(I^+) + m$, $\#(\bar{I}^+) = \#(\bar{I}^-) + m$. So there is complete symmetry between positive and negative m . Then without going into the constructive definition of $\Phi_\alpha^{(m)}(0)$ we can just accept the validity of (9.1) for all $m \in \mathbb{Z}$.

10. COMPARISON WITH KNOWN RESULTS

10.1. Lukyanov-Zamolodchikov formula. Consider the case $R = \infty$. Notice that $\Theta_\infty^{\text{sG}}(l, j | \alpha) = 0$, so, in the formula (8.2) the determinant contains a diagonal matrix. This means, in particular, that $\langle \mathbf{l}_{-2} \Phi_\alpha(0) \rangle_\infty^{\text{sG}} = 0$, and the first descendant with non-trivial one-point function is $\mathbf{l}_{-2} \bar{\mathbf{l}}_{-2} \Phi_\alpha(0)$ as it has been expected.

Let us consider the simplest ratio of one-point functions of primary fields. The formula (8.2) together with (1.13) gives:

$$\frac{\langle \beta_1^* \bar{\gamma}_1^* \Phi_\alpha^{(1)}(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} = \frac{1}{\nu} \left[M \frac{\sqrt{\pi} \Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})} \right]^{2\nu(\alpha-2)+2} \Gamma(\nu)^{-4(\alpha/2+(1-\nu)/2\nu)+2} \cot \frac{\pi}{2\nu} (\alpha\nu + 1).$$

So, comparing with the formulae (6.8) and (6.10) we obtain

$$\frac{\langle \Phi_{\alpha+2\frac{1-\nu}{\nu}}(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} = \left[M \frac{\sqrt{\pi} \Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})} \right]^{2(\nu\alpha+1-\nu)} H(\alpha/2 + (1-\nu)/2\nu),$$

where

$$H(x) = \frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)} \cdot \frac{\Gamma(x)}{\Gamma(x+1/2)} \cdot \frac{\Gamma(-x+1/2)}{\Gamma(-x)}.$$

This formula is in perfect agreement with the Lukyanov-Zamolodchikov formula [9] which reads in our notations as

$$\begin{aligned} \langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}} &= \left[M \frac{\sqrt{\pi} \Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})} \right]^{\frac{\nu^2 \alpha^2}{2(1-\nu)}} \\ &\times \exp \left(\int_0^\infty \left(\frac{\sinh^2(\nu \alpha t)}{2 \sinh(1-\nu)t \sinh t \cosh \nu t} - \frac{\nu^2 \alpha^2}{2(1-\nu)} e^{-2t} \right) \frac{dt}{t} \right). \end{aligned}$$

10.2. Fateev-Fradkin-Lukyanov-Zamolodchikov-Zamolodchikov formula. Let us consider the first non-trivial descendant for $R = \infty$, which is $\mathbf{l}_{-2} \bar{\mathbf{l}}_{-2} \Phi_\alpha(0)$. According to (5.1), (8.2) and (1.13) we have:

$$\begin{aligned} (10.1) \quad & (D_1(\alpha) D_1(2-\alpha))^2 \frac{\langle \mathbf{l}_{-2} \bar{\mathbf{l}}_{-2} \Phi_\alpha(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} \\ &= \left[M \frac{\sqrt{\pi} \Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})} \Gamma(\nu)^{-\frac{1}{\nu}} \right]^4 \frac{1}{\nu^2} \cot \frac{\pi}{2\nu} (\nu\alpha - 1) \cot \frac{\pi}{2\nu} (\nu\alpha + 1). \end{aligned}$$

Rewriting $D_1(\alpha) D_1(2-\alpha)$ by using (3.3) we obtain

$$\begin{aligned} \frac{\langle \mathbf{l}_{-2} \bar{\mathbf{l}}_{-2} \Phi_\alpha(0) \rangle_\infty^{\text{sG}}}{\langle \Phi_\alpha(0) \rangle_\infty^{\text{sG}}} &= - \left[M \frac{\sqrt{\pi} \Gamma(\frac{1}{2\nu})}{2\sqrt{1-\nu} \Gamma(\frac{1-\nu}{2\nu})} \right]^4 \\ &\times \frac{\Gamma(-\frac{1}{2} + \frac{\alpha}{2} + \frac{1}{2\nu}) \Gamma(\frac{1}{2} - \frac{\alpha}{2} + \frac{1}{2\nu}) \Gamma(1 - \frac{\alpha}{2} - \frac{1}{2\nu}) \Gamma(\frac{\alpha}{2} - \frac{1}{2\nu})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2} - \frac{1}{2\nu}) \Gamma(\frac{1}{2} + \frac{\alpha}{2} - \frac{1}{2\nu}) \Gamma(\frac{\alpha}{2} + \frac{1}{2\nu}) \Gamma(1 - \frac{\alpha}{2} + \frac{1}{2\nu})}. \end{aligned}$$

We want to compare this with the formula (1.8) of [2]. The parameters are identified as follows: $\xi = \frac{1-\nu}{\nu}$, $\eta = \alpha - 1$. Making this change of variables we find a perfect agreement.

10.3. Zamolodchikov formula. In [16] A. Zamolodchikov proves that for any two-dimensional Euclidian QFT on a cylinder the following formula holds:

$$(10.2) \quad \langle T_{z,z} T_{\bar{z},\bar{z}} \rangle = \langle T_{z,z} \rangle \langle T_{\bar{z},\bar{z}} \rangle - \langle T_{z,\bar{z}} \rangle^2.$$

Let us check that the formulae (8.2) agree with (10.2). We consider the sG theory with modified energy-momentum tensor. Obviously, considering (10.2) we have to set $\alpha = 0$, so, (10.2) reads

$$(10.3) \quad \langle \mathbf{l}_{-2} \bar{\mathbf{l}}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} = \langle \mathbf{l}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} \langle \bar{\mathbf{l}}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} - \left(2\pi\nu \frac{\mu^2}{\sin \pi\nu} \right)^2 \left(\langle \Phi_{2\frac{1-\nu}{\nu}} \rangle_R^{\text{sG}} \right)^2,$$

where the multiplier $2\pi\nu$ in the last term takes into account the CFT normalisation of the energy-momentum tensor and the scaling dimension of μ .

The case $\alpha = 0$ is special because the singularity of $\hat{R}(k, 0)$ at $k = 0$ cancels. From (7.4) we obtain

$$(10.4) \quad \Theta_R^{\text{sG}}(l, j|0) = \Theta_R^{\text{sG}}(-l, -j|0)$$

The formula (10.3) follows immediately from the particular cases of (8.2) :

$$(10.5) \quad \langle \mathbf{l}_{-2} \bar{\mathbf{l}}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} = \frac{M^4}{64\nu^2} \begin{vmatrix} \Theta_R^{\text{sG}}(\frac{i}{2\nu}, \frac{i}{2\nu}|0) & 2\pi\nu \cot \frac{\pi}{2\nu} + \Theta_R^{\text{sG}}(\frac{i}{2\nu}, -\frac{i}{2\nu}|0) \\ 2\pi\nu \cot \frac{\pi}{2\nu} + \Theta_R^{\text{sG}}(-\frac{i}{2\nu}, \frac{i}{2\nu}|0) & \Theta_R^{\text{sG}}(-\frac{i}{2\nu}, -\frac{i}{2\nu}|0) \end{vmatrix},$$

$$\langle \mathbf{l}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} = \frac{M^2}{8\nu} \Theta_R^{\text{sG}}(\frac{i}{2\nu}, \frac{i}{2\nu}|0), \quad \langle \bar{\mathbf{l}}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} = \frac{M^2}{8\nu} \Theta_R^{\text{sG}}(-\frac{i}{2\nu}, -\frac{i}{2\nu}|0),$$

$$2\pi\nu \frac{\mu^2}{\sin \pi\nu} \langle \Phi_{2\frac{1-\nu}{\nu}} \rangle_R^{\text{sG}} = \frac{M^2}{8\nu} (2\pi\nu \cot \frac{\pi}{2\nu} + \Theta_R^{\text{sG}}(\frac{i}{2\nu}, -\frac{i}{2\nu}|0)).$$

From (10.4) we have $\Theta_R^{\text{sG}}(\frac{i}{2\nu}, -\frac{i}{2\nu}|0) = \Theta_R^{\text{sG}}(-\frac{i}{2\nu}, \frac{i}{2\nu}|0)$.

On the other hand there is a simple independent way to compute the expectation values of the components of the energy-momentum tensor:

$$\begin{aligned} \langle \mathbf{l}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} &= \frac{1}{4} \left(-\frac{1}{R} + \frac{d}{dR} \right) E(R) + \frac{1}{2R} P(R) \\ \langle \bar{\mathbf{l}}_{-2} \cdot \mathbf{1} \rangle_R^{\text{sG}} &= \frac{1}{4} \left(-\frac{1}{R} + \frac{d}{dR} \right) E(R) - \frac{1}{2R} P(R) \\ 2\pi\nu \frac{\mu^2}{\sin \pi\nu} \langle \Phi_{2\frac{1-\nu}{\nu}} \rangle_R^{\text{sG}} &= -\frac{1}{4} \left(\frac{1}{R} + \frac{d}{dR} \right) E(R), \end{aligned}$$

where $E(R)$ and $P(R)$ are the energy and momentum of the ground state in Matsubara direction. Certainly, $P(R) = 0$ for the ground state, but we do not set it explicitly because the applications of our formulae are not really restricted to the ground state.

Let us see that they agree with the formulae (10.5). The check is based on two equations:

$$\begin{aligned}\frac{1}{i} \frac{\partial}{\partial R} \log \mathbf{a}(\zeta) &= 2\pi M (sh - R_{\text{dress}} * sh)(\zeta), \\ \frac{1}{i} \nu \zeta \frac{\partial}{\partial \zeta} \log \mathbf{a}(\zeta) &= 2\pi M R (ch - R_{\text{dress}} * ch)(\zeta),\end{aligned}$$

where $sh(\xi) = \frac{1}{2}(\xi^{\frac{1}{\nu}} - \xi^{-\frac{1}{\nu}})$, $ch(\xi) = \frac{1}{2}(\xi^{\frac{1}{\nu}} + \xi^{-\frac{1}{\nu}})$. Then using the formulae [15]

$$\begin{aligned}4E(R) &= 2\pi M^2 R \cot \frac{\pi}{2\nu} - \frac{M}{2\pi\nu} 2\text{Im} \int_0^\infty (\xi^{\frac{1}{\nu}} - \xi^{-\frac{1}{\nu}}) \log(1 + \mathbf{a}(\xi e^{+i0})) \frac{d\xi^2}{\xi^2}, \\ 4P(R) &= -\frac{M}{2\pi\nu} 2\text{Im} \int_0^\infty (\xi^{\frac{1}{\nu}} + \xi^{-\frac{1}{\nu}}) \log(1 + \mathbf{a}(\xi e^{+i0})) \frac{d\xi^2}{\xi^2}.\end{aligned}$$

one immediately proves that

$$\begin{aligned}\left(-\frac{1}{R} + \frac{d}{dR}\right) E(R) + \frac{2}{R} P(R) &= \frac{M^2}{2\nu} (e_+ * e_+ - e_+ * R_{\text{dress}} * e_+) \\ \left(-\frac{1}{R} + \frac{d}{dR}\right) E(R) - \frac{2}{R} P(R) &= \frac{M^2}{2\nu} (e_- * e_- - e_- * R_{\text{dress}} * e_-) \\ -\left(\frac{1}{R} + \frac{d}{dR}\right) E(R) &= \pi M^2 \cot \frac{\pi}{2\nu} + \frac{M^2}{2\nu} (e_+ * e_- - e_+ * R_{\text{dress}} * e_-),\end{aligned}$$

where $e_\pm(\xi) = \xi^{\pm\frac{1}{\nu}}$. These equations are obviously equivalent to the last three equations of (10.5).

11. CONCLUSIONS

The reader may notice certain discrepancy between simple final results of this paper and an indirect way of obtaining them in many cases. One may find the definition of the lattice regularisation of the temperature expectation values given in Section 2 to be especially hard to understand. This situation reminds the history of writing the exact formulae for the form factors in the sG model. They were given in [17], but the derivation was based on the quantum Gelfand-Levitan-Marchenko equations. These equations were derived using the lattice regularisation, and both their derivation and application are rather a matter of art than that of science. However, later in [18] it was explained that the same formulae can be derived starting from the bootstrap approach which follows from the first principles of QFT. We hope that the situation is similar in the present case. Let us summarise the situation and analyse it.

We start with CFT, and we want to describe the quotient spaces $\mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}}^{\text{quo}}$. We claim that this can be done using the operators $\beta^*(\lambda)$, $\gamma^*(\lambda)$, $\gamma_{\text{screen}}^*(\lambda)$ and creating these spaces starting from $\phi_\alpha(0)$. The three-point functions with primary fields $\phi_{1-\kappa}(-\infty)$, $\phi_{1+\kappa}(\infty)$ are described by functions $\omega^{\text{sc}}(\lambda, \mu|\kappa, \kappa, \alpha)$ and $\omega_0(\lambda/\mu, \alpha)$.

Moreover, the very meaning of our construction implies that similar formulae hold in the case when the primary fields $\phi_{1-\kappa}(-\infty)$, $\phi_{1+\kappa}(\infty)$ are replaced by any eigenvectors of the local integrals of motion. It must be possible to make these definitions directly in CFT without any reference to the lattice. This should be just a part of understanding the integrable structure of CFT.

The next statement concerns the integrable deformation of CFT. We claim that the local fields in the sG model are created by the operators

$$\begin{aligned}\beta^{+*}(\zeta) &= \beta^*(\mu\zeta) + \bar{\beta}_{\text{screen}}^*(\zeta/\mu), \\ \gamma^{+*}(\zeta) &= \gamma^*(\mu\zeta) + \bar{\gamma}_{\text{screen}}^*(\zeta/\mu), \\ \beta^{-*}(\zeta) &= \bar{\beta}^*(\zeta/\mu) + \beta_{\text{screen}}^*(\mu\zeta), \\ \gamma^{-*}(\zeta) &= \bar{\gamma}^*(\zeta/\mu) + \gamma_{\text{screen}}^*(\mu\zeta),\end{aligned}$$

which provide a basis of local fields consistent with CFT (without finite counterterms). The one-point functions of these operators are computed using $\omega_R^{\text{sG}}(\zeta, \xi|\alpha)$ and $\omega_0(\zeta/\xi, \alpha)$. It must be possible to explain this fact without reference to the lattice, by proper understanding of an integrable deformation of CFT in its integrable formulation.

APPENDIX A. THREE POINT FUNCTION FROM CFT

The three-point functions in CFT with $c < 1$ were computed by Dotsenko and Fateev [19]. Here it will be convenient for us to use the nice formula obtained for Liouville theory by Zamolodchikov and Zamolodchikov [20]. We shall only need not the three-point function itself but rather its ratio to the one with a shifted parameter. Teschner [21] proved the formula of [20] computing exactly this kind of ratio. Another remark is in order here. We apply the formulae obtained for Liouville theory to $c < 1$ model. This procedure involves some problems because of difficulties with understanding the basic function $\Upsilon(x)$ for $b^2 < 0$ (the notation is given later). However, these problems do not concern the ratios of three-point functions which we consider, and which are expressed in terms of Γ -functions. Finally, we normalise the primary fields as Liouville exponential fields, i.e. they have non-trivial constant in the two-point function. For the last point, we refer the reader to Appendix C of [22]. We change the normalisation of the cosmological constant of Liouville model comparing to [20]:

$$\mathcal{A}^{\text{Lv}} = \int \left\{ \frac{1}{4\pi} \partial_z \varphi(z, \bar{z}) \partial_{\bar{z}} \varphi(z, \bar{z}) + \frac{\mu^2}{\sin \pi b^2} e^{b\varphi(z, \bar{z})} \right\} \frac{idz \wedge d\bar{z}}{2},$$

Consider the three-point function of exponential fields. In our cylindrical coordinates (recall that the map to the Riemann sphere is given by $z \rightarrow e^{-z}$) it reads

$$(A.1) \quad \langle e^{a_1 \varphi(-\infty)} e^{a_2 \varphi(z, \bar{z})} e^{a_3 \varphi(\infty)} \rangle = e^{(\Delta_{a_3} - \Delta_{a_1})(z + \bar{z})} C(a_1, a_2, a_3),$$

where

$$\Delta_a = a(Q - a), \quad Q = b + 1/b.$$

We have from [20]

$$C(a, Q/2 - k, Q/2 + k) = (\mu^2 \Gamma(1 + b^2)^2 b^{-2-2b^2})^{-a/b} \times \frac{\Upsilon_0 \Upsilon(2a) \Upsilon(Q - 2k) \Upsilon(Q + 2k)}{\Upsilon^2(a) \Upsilon(a - 2k) \Upsilon(a + 2k)}.$$

Following [20] we use the function $\Upsilon(x)$ defined by the equation

$$\frac{\Upsilon(x + b)}{\Upsilon(x)} = \gamma(bx) b^{1-2bx}, \quad \frac{\Upsilon(x + 1/b)}{\Upsilon(x)} = \gamma(x/b) b^{-1+2x/b},$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, as well as the constant $\Upsilon_0 = (d\Upsilon/dx)(0)$.

An important point is that the exponent of the primary field must be of the sign opposite to that of the exponent in the Lagrangian. So, we consider the shift:

$$(A.2) \quad \frac{C(a - b, Q/2 - k, Q/2 + k)}{C(a, Q/2 - k, Q/2 + k)} = \mu^2 \Gamma(1 + b^2)^2 \frac{\gamma^2(ab - b^2)}{\gamma(2ab - 2b^2) \gamma(2ab - b^2)} \times \gamma(ab - b^2 - 2kb) \gamma(ab - b^2 + 2kb).$$

Now we go to our case of “complex Liouville theory” by the substitution

$$b^2 = \nu - 1, \quad 2ab = \nu\alpha, \quad 2kb = \nu\kappa,$$

and obtain (6.3) in Section 6.

APPENDIX B. ASYMPTOTIC EXPANSION OF Θ

The first few terms of the asymptotic expansion of the function $\Theta(l, m|\kappa, \alpha)$ as $\kappa \rightarrow \infty$ are given as follows. We set $p = \kappa/\sqrt{8(1-\nu)}$,

$$\begin{aligned} \Theta(il, im|\kappa, \alpha) &= \Theta^{\text{even}}(il, im|\kappa, \alpha) + \Theta^{\text{odd}}(il, im|\kappa, \alpha) d_\alpha, \\ \Theta^*(il, im|\kappa, \alpha) &= \sum_{n=0}^{\infty} \Theta_{2n}^*(il, im|\kappa, \alpha) p^{-2n}, \end{aligned}$$

where

$$d_\alpha = \frac{\nu(\nu - 2)}{\nu - 1}(\alpha - 1).$$

$$\begin{aligned}
\Theta_0^{\text{even}}(il, im|\kappa, \alpha) &= -\frac{1}{l+m}, \\
\Theta_2^{\text{even}}(il, im|\kappa, \alpha) &= \frac{\Delta_\alpha}{24\nu} + \frac{2[l+m]\nu-1}{48\nu}, \\
\Theta_4^{\text{even}}(il, im|\kappa, \alpha) &= \frac{\Delta_\alpha^2}{1152\nu^3} \left(-[l+m]\nu+1 \right) \\
&\quad - \frac{\Delta_\alpha}{5760(\nu-1)\nu^3} \left([2(7l^2+8ml+7m^2)+4(l+m)]\nu^3 \right. \\
&\quad \left. + [2(7l^2+8ml+7m^2)+27(l+m)+4]\nu^2 - [27(l+m)+16]\nu+16 \right) \\
&\quad - \frac{1}{23040(\nu-1)\nu^3} \left([28(l+m)^3+8(l^2+3ml+m^2)]\nu^4 \right. \\
&\quad \left. + [28(l+m)^3+4(20l^2+39ml+20m^2)+16(l+m)]\nu^3 \right. \\
&\quad \left. - [4(20l^2+39ml+20m^2)+69(l+m)+6]\nu^2 + [69(l+m)+18]\nu-18 \right), \\
\Theta_6^{\text{even}}(il, im|\kappa, \alpha) &= \frac{\Delta_\alpha^3}{82944\nu^5} \left([(l+m)^2]\nu^2 - 3[l+m]\nu+2 \right) \\
&\quad + \frac{\Delta_\alpha^2}{5806080(\nu-1)\nu^5} \left([126(l+m)(3l^2+2ml+3m^2)+8(13l^2+50ml+13m^2)]\nu^4 \right. \\
&\quad \left. - [126(3l^2+2ml+3m^2)(l+m)+9(143l^2+270ml+143m^2)+408(l+m)]\nu^3 \right. \\
&\quad \left. + [9(143l^2+270ml+143m^2)+1773(l+m)+256]\nu^2 \right. \\
&\quad \left. - [1773(l+m)+774]\nu+774 \right) \\
&\quad + \frac{\Delta_\alpha}{3870720(\nu-1)^2\nu^5} \left([4(93l^4+208ml^3+310m^2l^2+208m^3l+93m^4) \right. \\
&\quad \left. + 328(l+m)(l^2+ml+m^2)+32(3l^2+5ml+3m^2)]\nu^6 \right. \\
&\quad \left. - [8(93l^4+208ml^3+310m^2l^2+208m^3l+93m^4) \right. \\
&\quad \left. + 4(610l^2+561ml+610m^2+68)(l+m)+8(183l^2+274ml+183m^2)]\nu^5 \right. \\
&\quad \left. + [4(93l^4+208ml^3+310m^2l^2+208m^3l+93m^4) \right. \\
&\quad \left. + 8(l+m)(528l^2+479ml+528m^2) \right. \\
&\quad \left. + 5831l^2+8422ml+5831m^2+2138(l+m)+184]\nu^4 \right. \\
&\quad \left. - [4(528l^2+479ml+528m^2+1504)(l+m)+2(4367l^2+6230ml+4367m^2) \right. \\
&\quad \left. + 1104]\nu^3 + [4367l^2+6230ml+4367m^2+7772(l+m)+2556]\nu^2 \right. \\
&\quad \left. - [3886(l+m)+2904]\nu+1452 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{23224320(\nu-1)^2\nu^5} \left([744(l+m)^5 + 16(41l^4 + 205ml^3 + 308m^2l^2 + 205m^3l + 41m^4) \right. \\
& \quad + 192(l^2 + 4ml + m^2)(l+m)]\nu^7 \\
& \quad - [1488(l+m)^5 + 4(1477l^4 + 5990ml^3 + 8986m^2l^2 + 5990m^3l + 1477m^4) \\
& \quad + 24(163l^2 + 413ml + 163m^2)(l+m) + 96(9l^2 + 25ml + 9m^2)]\nu^6 \\
& \quad + [744(l+m)^5 + 8(1313l^4 + 5170ml^3 + 7754m^2l^2 + 5170m^3l + 1313m^4) \\
& \quad + 6(2953l^2 + 6064ml + 2953m^2 + 184)(l+m) + 4(2023l^2 + 4538ml + 2023m^2)]\nu^5 \\
& \quad - [4(1313l^4 + 5170ml^3 + 7754m^2l^2 + 5170m^3l + 1313m^4) \\
& \quad + 6(l+m)(4602l^2 + 8824ml + 4602m^2 + 1125) \\
& \quad + 24755l^2 + 50326ml + 24755m^2 + 360]\nu^4 \\
& \quad + [6(2301l^2 + 4412ml + 2301m^2 + 2460)(l+m) \\
& \quad + 2(16663l^2 + 32174ml + 16663m^2) + 1800]\nu^3 \\
& \quad - [16663l^2 + 32174ml + 16663m^2 + 18180(l+m) + 3600]\nu^2 \\
& \quad \left. + [9090(l+m) + 3600]\nu - 1800 \right),
\end{aligned}$$

$$\Theta_0^{\text{odd}}(il, im|\kappa, \alpha) = 0,$$

$$\Theta_2^{\text{odd}}(il, im|\kappa, \alpha) = 0,$$

$$\Theta_4^{\text{odd}}(il, im|\kappa, \alpha) = -\Delta_\alpha \frac{l-m}{2880\nu^2},$$

$$\Theta_6^{\text{odd}}(il, im|\kappa, \alpha) = \Delta_\alpha^2 \frac{l-m}{69120\nu^4} ([l+m]\nu - 2)$$

$$\begin{aligned}
& + \Delta_\alpha \frac{l-m}{2903040(\nu-1)\nu^4} \left((2[41l^2 + 42ml + 41m^2] + 48(l+m))\nu^3 \right. \\
& \quad \left. - [2(41l^2 + 42ml + 41m^2) + 285(l+m) + 96]\nu^2 + [285(l+m) + 302]\nu - 302 \right).
\end{aligned}$$

Expectation values of the fermions β_{2j-1}^* , etc. are calculated by appropriate specialisation of l, m . We give below the results upto degree κ^{-8} in the case

$$\begin{aligned}
\Omega_{2r-1, 2s-1}(\kappa, \alpha) &= -\frac{r+s-1}{\nu} (\sqrt{2}p\nu)^{2r+2s-2} \Theta \left(\frac{i(2r-1)}{2\nu}, \frac{i(2s-1)}{2\nu} \middle| \kappa, \alpha \right) \\
&= \Omega_{2r-1, 2s-1}^{\text{even}}(\kappa, \alpha) + \Omega_{2r-1, 2s-1}^{\text{odd}}(\kappa, \alpha) d_\alpha.
\end{aligned}$$

In the next formulae, $I_{2n-1}(\kappa)$'s denote the vacuum eigenvalues of the integrals of motion which can be found for instance in [13].

$$\begin{aligned}
\Omega_{1,1}^{\text{even}}(\kappa, \alpha) &= I_1(\kappa) - \frac{\Delta_\alpha}{12}, \\
\Omega_{1,3}^{\text{even}}(\kappa, \alpha) &= I_3(\kappa) - \frac{\Delta_\alpha}{6} I_1(\kappa) + \frac{\Delta_\alpha^2}{144} + \frac{c+5}{1080} \Delta_\alpha, \\
\Omega_{1,3}^{\text{odd}}(\kappa, \alpha) &= -\frac{\Delta_\alpha}{360},
\end{aligned}$$

$$\begin{aligned}
\Omega_{1,5}^{\text{even}}(\kappa, \alpha) &= I_5(\kappa) - \frac{\Delta_\alpha}{4} I_3(\kappa) + \left(\frac{\Delta_\alpha^2}{48} + \frac{c+11}{360} \Delta_\alpha \right) I_1(\kappa) \\
&\quad - \frac{\Delta_\alpha^3}{1728} - \frac{13(c+35)}{90720} \Delta_\alpha^2 - \frac{2c^2+21c+70}{60480} \Delta_\alpha, \\
\Omega_{3,3}^{\text{even}}(\kappa, \alpha) &= I_5(\kappa) - \frac{\Delta_\alpha}{4} I_3(\kappa) + \left(\frac{\Delta_\alpha^2}{48} + \frac{c+2}{360} \Delta_\alpha + \frac{c+2}{1440} \right) I_1(\kappa) \\
&\quad - \frac{1}{1728} \Delta_\alpha^3 - \frac{5c-14}{18144} \Delta_\alpha^2 - \frac{10c^2+37c+70}{362880} \Delta_\alpha - \frac{1/2c^2+c}{36288}, \\
\Omega_{1,5}^{\text{odd}}(\kappa, \alpha) &= -\frac{\Delta_\alpha^2}{1440} - \frac{c+7}{7560} \Delta_\alpha - \frac{\Delta_\alpha}{120} I_1(\kappa).
\end{aligned}$$

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